Introduction to the Functional RG and Applications to Gauge Theories

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Prologue

This lecture course[1] is intended to fill the gap between graduate courses on quantum field theory and specialized reviews or forefront-research articles on functional renormalization group approaches to quantum field theory and gauge theories.

These lecture notes are meant for advanced students who want to get acquainted with modern renormalization group (RG) methods as well as functional approaches to quantum gauge theories. In the first lecture, the functional renormalization group is introduced with a focus on the flow equation for the effective average action. The second lecture is devoted to a discussion of flow equations and symmetries in general, and flow equations and gauge symmetries in particular. The third lecture deals with the flow equation in the background formalism which is particularly convenient for analytical computations of truncated flows. The fourth lecture concentrates on the transition from microscopic to macroscopic degrees of freedom; even though this is discussed here in the language and the context of QCD, the developed formalism is much more general and will be useful also for other systems. Sections which have an asterisk $*$ in the section title contain more advanced material and may be skipped during a first reading.

This is not a review. I apologize for many omissions of further interesting and important aspects of this field (and their corresponding references). General reviews and more complete reference lists can be found in [1, 2, 3, 4, 5, 6, 7, 8]. A guide to more specialized literature is given in the Further-Reading subsections at the end of some sections.

1 Introduction

Quantum and statistical field theory are two sides of the same medal, representing the fundament on which modern physics is built. Both branches have been molded by the concept of the renormalization group. The renormalization

[1] Lectures held at the 2006 ECT* School "Renormalization Group and Effective Field Theory Approaches to Many-Body Systems", Trento, Italy.

group deals with the physics of scales. A central theme is the understanding of the macroscopic physics at long distances or low momenta in terms of the fundamental microscopic interactions. Bridging this gap from micro to macro scales requires a thorough understanding of quantum or statistical fluctuations on all the scales in between.

Field theories with gauge symmetry are of central importance, in particular, since all elementary particle-physics interactions are described by gauge theories. Moreover, nonabelian gauge theories – and most prominently quantum chromodynamics (QCD) – are in many respects paradigmatic, since they exhibit numerous features that are encountered in various field theoretical systems both in particle physics as well as condensed-matter systems. During the transition from micro to macro scales, these theories turn from weak to strong coupling, the relevant degrees of freedom are changed, the realization of the fundamental symmetries is different on the various scales, and the phase diagram is expected to have a rich structure, being formed by different and competing collective phenomena.

A profound understanding of gauge theories thus requires not just one but a whole toolbox of field theoretical methods. In addition to analytical perturbative methods for weak coupling and numerical lattice gauge theory for arbitrary couplings, *functional methods* begin to bridge the gap, since they are not restricted to weak coupling and can still largely be treated analytically. Functional methods aim at the computation of generating functionals of correlation functions, such as the effective action that governs the dynamics of the macroscopic expectation values of the fields. These generating functionals contain all relevant physical information about a theory, once the microscopic fluctuations have been integrated out.

The functional RG combines this functional approach with the RG idea of treating the fluctuations not all at once but successively from scale to scale [9, 10]. Instead of studying correlation functions after having averaged over all fluctuations, only the *change* of the correlation functions as induced by an infinitesimal momentum shell of fluctuations is considered. From a structural viewpoint, this allows to transform the functional-integral structure of standard field theory formulations into a functional differential structure [11, 12, 13, 14]. This goes along not only with a better analytical and numerical accessibility and stability, but also with a great flexibility of devising approximations adapted to a specific physical system. In addition, structural investigations of field theories from first principles such as proofs of renormalizability can more elegantly and efficiently be performed with this strategy [13, 15, 16, 17].

The central tool of the functional RG is given by a flow equation. This flow equation describes the evolution of correlation functions or their generating functional under the influence of fluctuations. It connects a well-defined initial quantity, e.g., the microscopic correlation functions in a perturbative domain, in an exact manner with the desired full correlation functions after having integrated out the fluctuations. Hence, solving the flow equation corresponds to solving the full theory.

The complexity of quantum gauge theories and QCD are a serious challenge for all field theoretical methods. The construction of flow equations for gauge theories has to take special care of the gauge symmetry, i.e., the invariance of the theory under *local* transformations of the fields in coordinate space [18, 19, 20, 21, 22]. The beauty of gauge symmetry is turned into the beast of complex dynamical equations and nontrivial symmetry constraints both of which have to be satisfied by the flow of the correlation functions. Nevertheless, the success of the functional RG in many branches of physics as described in other lectures of this volume makes it a promising tool also for gauge theories. The recent rapid development of functional methods and their application to gauge theories and QCD also in the strong-coupling domain confirm this expectation. In combination with and partly complementary to other field theoretical methods, the functional RG has the potential to shed light on some of the still hardly accessible parameter regions of quantum gauge theories.

2 Functional RG Approach to Quantum Field Theory

2.1 Basics of QFT

In quantum field theory (QFT), all physical information is stored in correlation functions. For instance, consider a collider experiment with two incident beams and $(n-2)$ scattering products. All information about this process can be obtained from the *n-point function*, a correlator of n quantum fields. In QFT, we obtain this correlator by definition from the product of n field operators at different spacetime points $\varphi(x_n)$ averaged over all possible field configurations (quantum fluctuations).

In Euclidean QFT, the field configurations are weighted with an exponential of the action $S[\varphi]$,

$$\langle \varphi(x_1) \dots \varphi(x_n) \rangle := \mathcal{N} \int \mathcal{D}\varphi \, \varphi(x_1) \dots \varphi(x_n) \, \mathrm{e}^{-S[\varphi]}, \tag{1}$$

where we fix the normalization $\mathcal{N}$ by demanding that $\langle 1 \rangle = 1$. We assume that Minkowski-valued correlators can be defined from the Euclidean ones by analytic continuation. We also assume that a proper regularized definition of the measure can be given (for instance, using a spacetime lattice discretization), which we formally write as $\int \mathcal{D}\varphi \to \int_\Lambda \mathcal{D}\varphi$; here, Λ denotes an ultraviolet (UV) cutoff. This regularized measure should also preserve the symmetries of the theory: for a symmetry transformation U which acts on the fields, $\varphi \to \varphi^U$, and leaves the action invariant, $S[\varphi] \to S[\varphi^U] \equiv S[\varphi]$, the invariance of the measure implies

$$\int_\Lambda \mathcal{D}\varphi \to \int_\Lambda \mathcal{D}\varphi^U \equiv \int_\Lambda \mathcal{D}\varphi. \tag{2}$$

For simplicity, let φ denote a real scalar field; the following discussion also holds for other fields such as fermions with minor modifications. All n-point correlators are summarized by the generating functional $Z[J]$,

$$Z[J] \equiv \mathrm{e}^{W[J]} = \int \mathcal{D}\varphi \, \mathrm{e}^{-S[\varphi]+\int J\varphi}, \tag{3}$$

with source term $\int J\varphi = \int \mathrm{d}^D x \, J(x)\varphi(x)$. All n-point functions are obtained by functional differentiation:

$$\langle \varphi(x_1)\dots\varphi(x_n)\rangle = \frac{1}{Z[0]}\left(\frac{\delta^n Z[J]}{\delta J(x_1)\dots\delta J(x_n)}\right)_{J=0}. \tag{4}$$

Once the generating functional is computed, the theory is solved.

In Eq. (3), we have also introduced the generating functional of *connected correlators*[2], $W[J] = \ln Z[J]$, which, loosely speaking, is a more efficient way to store the physical information. An even more efficient information storage is obtained by a Legendre transform of $W[J]$: the *effective action* Γ:

$$\Gamma[\phi] = \sup_J \left(\int J\phi - W[J]\right). \tag{5}$$

For any given ϕ, a special $J \equiv J_{\mathrm{sup}} = J[\phi]$ is singled out for which $\int J\phi - W[J]$ approaches its supremum. Note that this definition of Γ automatically guarantees that Γ is convex. At $J = J_{\mathrm{sup}}$, we get

$$\begin{aligned} 0 &\overset{!}{=} \frac{\delta}{\delta J(x)}\left(\int J\phi - W[J]\right) \\ \Rightarrow \quad \phi &= \frac{\delta W[J]}{\delta J} = \frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J} = \langle\varphi\rangle_J. \end{aligned} \tag{6}$$

This implies that ϕ corresponds to the expectation value of φ in the presence of the source J. The meaning of Γ becomes clear by studying its derivative at $J = J_{\mathrm{sup}}$

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = -\int_y \frac{\delta W[J]}{\delta J(y)}\frac{\delta J(y)}{\delta\phi(x)} + \int_y \frac{\delta J(y)}{\delta\phi(x)}\phi(y) + J(x) \overset{(6)}{=} J(x). \tag{7}$$

This is the *quantum equation of motion* by which the effective action $\Gamma[\phi]$ governs the dynamics of the field expectation value, taking the effects of all quantum fluctuations into account.

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[2] In this short introduction, we use but make no attempt at fully explaining the standard QFT nomenclature; for the latter, we refer the reader to any standard QFT textbook, such as [23, 24].

From the definition of the generating functional, we can straightforwardly derive an equation for the effective action:

$$\mathrm{e}^{-\Gamma[\phi]} = \int_\Lambda \mathcal{D}\varphi \, \exp\left(-S[\phi+\varphi] + \int \frac{\delta\Gamma[\phi]}{\delta\phi}\,\varphi\right). \tag{8}$$

Here, we have performed a shift of the integration variable, $\varphi \to \varphi + \phi$. We observe that the effective action is determined by a nonlinear first-order functional differential equation, the structure of which is itself a result of a functional integral. An exact determination of $\Gamma[\phi]$ and thus an exact solution has so far been found only for rare, special cases.

As a first example of a functional technique, a solution of Eq. (8) can be attempted by a *vertex expansion* of $\Gamma[\phi]$,

$$\Gamma[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^D x_1 \ldots \mathrm{d}^D x_n \, \Gamma^{(n)}(x_1,\ldots,x_n)\, \phi(x_1)\ldots\phi(x_n), \tag{9}$$

where the expansion coefficients $\Gamma^{(n)}$ correspond to the *one-particle irreducible (1PI) proper vertices.* Inserting Eq. (9) into Eq. (8) and comparing the coefficients of the field monomials results in an infinite tower of coupled integro-differential equations for the $\Gamma^{(n)}$: the Dyson-Schwinger equations. This functional method of constructing approximate solutions to the theory via truncated Dyson-Schwinger equations, i.e., via a finite truncation of the series Eq. (9) has its own merits and advantages; their application to gauge theories is well developed; see, e.g., [25, 26, 27, 28]. Here, we proceed by amending the RG idea to functional techniques in QFT.

2.2 RG Flow equation

A versatile approach to the computation of Γ is based on RG concepts [14]. Whereas a computation via Eq. (8) or via Dyson-Schwinger equations corresponds to integrating-out all fluctuations at once, we can implement Wilson's idea of integrating out modes momentum shell by momentum shell.

In terms of Γ, we are looking for an interpolating action Γ_k, which is also called *effective average action*, with a momentum-shell parameter k, such that Γ_k for $k \to \Lambda$ corresponds to the bare action to be quantized; the full quantum action Γ should be approached for $k \to 0$,

$$\Gamma_{k\to\Lambda} \simeq S_{\text{bare}}, \quad \Gamma_{k\to 0} = \Gamma. \tag{10}$$

This can indeed be constructed from the generating functional. For this, let us define the IR regulated functional

$$\begin{aligned}\mathrm{e}^{W_k[J]} \equiv Z_k[J] &:= \exp\left(-\Delta S_k\left[\frac{\delta}{\delta J}\right]\right) Z[J] \\ &= \int_\Lambda \mathcal{D}\varphi \, \mathrm{e}^{-S[\varphi]-\Delta S_k[\varphi]+\int J\varphi},\end{aligned} \tag{11}$$

where

$$\Delta S_k[\varphi] = \frac{1}{2}\int \frac{\mathrm{d}^D q}{(2\pi)^D}\, \varphi(-q) R_k(q) \varphi(q) \tag{12}$$

is a regulator term which is quadratic in φ and can be viewed as a momentum-dependent mass term. The regulator function $R_k(q)$ should satisfy

$$\lim_{q^2/k^2 \to 0} R_k(q) > 0, \tag{13}$$

which implements an IR regularization. For instance, if $R_k \sim k^2$ for $q^2 \ll k^2$, the regulator screens the IR modes in a mass-like fashion, $m^2 \sim k^2$. Furthermore,

$$\lim_{k^2/q^2 \to 0} R_k(q) = 0, \tag{14}$$

which implies that the regulator vanishes for $k \to 0$. As an immediate consequence, we automatically recover the standard generating functional as well as the full effective action in this limit: $Z_{k\to 0}[J] = Z[J]$ and $\Gamma_{k\to 0} = \Gamma$. The third condition is

$$\lim_{k^2 \to \Lambda \to \infty} R_k(q) \to \infty, \tag{15}$$

which induces that the functional integral is dominated by the stationary point of the action in this limit. This justifies the use of a saddle-point approximation which filters out the classical field configuration and the bare action, $\Gamma_{k\to\Lambda} \to S + \mathrm{const.}$. A sketch of a typical regulator that satisfies these three requirements is shown in Fig. 1. Incidentally, the regulator is frequently written as

$$R_k(p^2) = p^2\, r(p^2/k^2), \tag{16}$$

where $r(y)$ is a dimensionless regulator shape function with a dimensionless momentum argument. The requirements (13)-(15) translate in a obvious manner into corresponding requirements for $r(y)$.

Since we already know that the interpolating functional Γ_k exhibits the correct limits, let us now study the intermediate trajectory. We start with the generating functional $W_k[J]$, using the abbreviations

$$t = \ln \frac{k}{\Lambda}, \qquad \partial_t = k\frac{\mathrm{d}}{\mathrm{d}k}. \tag{17}$$

Keeping the source J fixed, i.e., k independent, we obtain

$$\begin{aligned}\partial_t W_k[J] &= -\frac{1}{2}\int \mathcal{D}\varphi\, \varphi(-q)\, \partial_t R_k(q)\, \varphi(q) \mathrm{e}^{-S-\Delta S+\int J\varphi} \\ &= -\frac{1}{2}\int \frac{\mathrm{d}^D q}{(2\pi)^D}\, \partial_t R_k(q)\, G_k(q) + \partial_t \Delta S_k[\phi]. \end{aligned} \tag{18}$$

Here, we have defined the *connected* propagator

$$G_k(p) = \left(\frac{\delta^2 W_k}{\delta J \delta J}\right)(p) = \langle \varphi(-p)\varphi(p)\rangle - \langle\varphi(-p)\rangle\langle\varphi(p)\rangle. \tag{19}$$

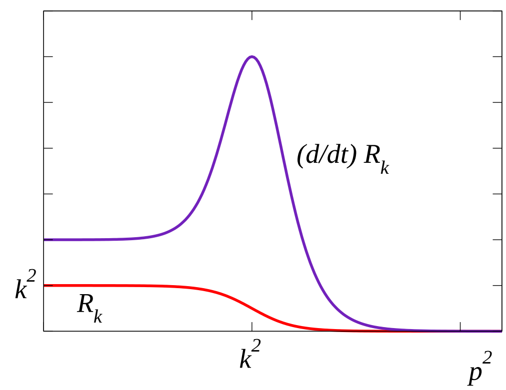


Fig. 1. Sketch of a regulator function $R_k(p^2)$ (lower curve) and its derivative $\partial_t R_k(p^2)$ (upper curve). Whereas the regulator provides for an IR regularization for all modes with $p^2 \lesssim k^2$, its derivative implements the Wilsonian idea of integrating out fluctuations within a momentum shell near $p^2 \simeq k^2$.

(Note that we frequently change from coordinate to momentum space or vice versa by Fourier transformation for reasons of convenience.) Now, we are in a position to define the interpolating effective action Γ_k by a slightly modified Legendre transform,[3]

$$\Gamma_k[\phi] = \sup_J \left(\int J\phi - W_k[J] \right) - \Delta S_k[\phi]. \tag{20}$$

Since we later want to study Γ_k as a functional of a k-independent field ϕ, it is clear from Eq. (20) that the source $J \equiv J_{\text{sup}} = J[\phi]$ for which the supremum is approached is necessarily k dependent. As before, we get at $J = J_{\text{sup}}$:

$$\phi(x) = \langle \varphi(x) \rangle_J = \frac{\delta W_k[J]}{\delta J(x)}. \tag{21}$$

The quantum equation of motion receives a regulator modification,

$$J(x) = \frac{\delta \Gamma_k[\phi]}{\delta \phi(x)} + \big(R_k \phi\big)(x). \tag{22}$$

From this, we deduce[4]:

$$\frac{\delta J(x)}{\delta \phi(y)} = \frac{\delta^2 \Gamma_k[\phi]}{\delta \phi(x) \delta \phi(y)} + R_k(x,y). \tag{23}$$

[3] Now, only the "sup" part of Γ_k is convex. For finite k, any non-convexity of Γ_k must be of the form of the last regulator term of Eq. (20).

[4] In case of fermionic Grassmann-valued fields, the following ϕ derivative should act on Eq. (22) from the right.

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\,\partial_t R_k$$

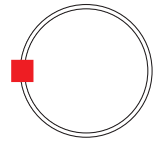


Fig. 2. Diagrammatic representation of the flow equation (28): the flow of Γ_k is given by a one-loop form, involving the full propagator $G_k = (\Gamma_k^{(2)} + R_k)^{-1}$ (double line) and an operator insertion in the form of $\partial_t R_k$ (filled box).

On the other hand, we obtain from Eq. (21):

$$\frac{\delta\phi(y)}{\delta J(x')} = \frac{\delta^2 W_k[J]}{\delta J(x')\delta J(y)} \equiv G_k(y-x'). \tag{24}$$

This implies the important identity

$$\begin{aligned}\delta(x-x') = \frac{\delta J(x)}{\delta J(x')} &= \int \mathrm{d}^D y\, \frac{\delta J(x)}{\delta\phi(y)}\, \frac{\delta\phi(y)}{\delta J(x')}\\ &= \int \mathrm{d}^D y\, (\Gamma_k^{(2)}[\phi] + R_k)(x,y) G_k(y-x'),\end{aligned} \tag{25}$$

or, in operator notation,

$$\mathbb{1} = (\Gamma_k^{(2)} + R_k)\, G_k. \tag{26}$$

Here, we have introduced the short-hand notation

$$\Gamma_k^{(n)}[\phi] = \frac{\delta^n \Gamma_k[\phi]}{\delta\phi \dots \delta\phi}. \tag{27}$$

Collecting all ingredients, we can finally derive the flow equation for Γ_k for fixed ϕ and at $J = J_{\mathrm{sup}}$ [14]:

$$\begin{aligned}\partial_t \Gamma_k[\phi] &= -\partial_t W_k[J]|_\phi + \int (\partial_t J)\phi - \partial_t \Delta S_k[\phi] = -\partial_t W_k[J]|_J - \partial_t \Delta S_k[\phi]\\ &\overset{(18)}{=} \frac{1}{2} \int \frac{\mathrm{d}^D q}{(2\pi)^D}\, \partial_t R_k(q)\, G_k(q)\\ &\overset{(26)}{=} \frac{1}{2}\, \mathrm{Tr}\left[\partial_t R_k \left(\Gamma_k^{(2)}[\phi] + R_k\right)^{-1}\right].\end{aligned} \tag{28}$$

This flow equation forms the starting point of all our further investigations. Hence, let us carefully discuss a few of its properties, as they are apparent already at this stage:

- The flow equation is a functional differential equation for Γ_k. In contrast to Eq. (8), no functional integral has to be performed to reveal the full structure of the equation.

- We have *derived* the flow equation from the standard starting point of QFT: the generating functional. But a different – if not inverse – perspective is also legitimate. We may *define* QFT based on the flow equation. For given suitable initial conditions, for instance, by defining the bare action at a high UV cutoff scale $k = \Lambda$, the flow equation defines a trajectory to the full quantum theory described by the full effective action Γ. In the case of additional symmetries, the QFT-defining flow equation may be supplemented by symmetry constraints to the effective action.
- The purpose of the regulator is actually twofold: by construction, the occurrence of R_k in the denominator of Eq. (28) guarantees the IR regularization by construction. In addition to this and thanks to the conditions (13) and (14), the derivative $\partial_t R_k$ occurring in the numerator of Eq. (28) ensures also UV regularization, since its predominant support lies on a smeared momentum shell near $p^2 \sim k^2$. A typical shape of the regulator and its derivative is depicted in Fig. 1. The peaked structure of $\partial_t R_k$ implements nothing but the Wilsonian idea of integrating over momentum shells and implies that the flow is localized in momentum space.
- The solution to the flow equation (28) corresponds to an RG trajectory in *theory space*. The latter is a space of all action functionals spanned by all possible invariant operators of the field. The two ends of the trajectory are given by the initial condition $\Gamma_{k\to\Lambda} = S_{\text{bare}}$, and the full effective action $\Gamma_{k\to 0} = \Gamma$.
- Apart from the conditions (13)-(15), the regulator can be chosen arbitrarily. Of course, the precise form of the trajectory depends on the regulator R_k. The variation of the trajectory with respect to R_k reflects the RG scheme dependence of a non-universal quantity; see Fig. 3. Nevertheless, the final point on the trajectory is independent of R_k as is guaranteed by Eqs. (13)-(15).
- The flow equation has a one-loop structure, but is nevertheless an exact equation, as is signaled by the occurrence of the exact propagator in the loop; see Fig. 2. The one-loop structure is a direct consequence of ΔS_k being quadratic in the field operator φ which is coupled to the source [29].
- Perturbation theory can immediately be re-derived from the flow equation. For instance, imposing the loop expansion on Γ_k, $\Gamma_k = S + \hbar \Gamma_k^{1-\text{loop}} + \mathcal{O}(\hbar^2)$, it becomes obvious that, to one-loop order, $\Gamma_k^{(2)}$ can be replaced by $S^{(2)}$ on the right-hand side of Eq. (28). From this, we infer:

$$\begin{aligned} \partial_t \Gamma_k^{1-\text{loop}} &= \frac{1}{2}\,\mathrm{Tr}\left[\partial_t R_k \left(S^{(2)} + R_k\right)^{-1}\right] = \frac{1}{2}\,\partial_t\,\mathrm{Tr}\,\ln(S^{(2)} + R_k) \\ &\Rightarrow \Gamma^{1-\text{loop}} = S + \frac{1}{2}\,\mathrm{Tr}\,\ln S^{(2)} + \text{const.} \end{aligned}$$

The last formula corresponds to the standard one-loop effective action, as it should.

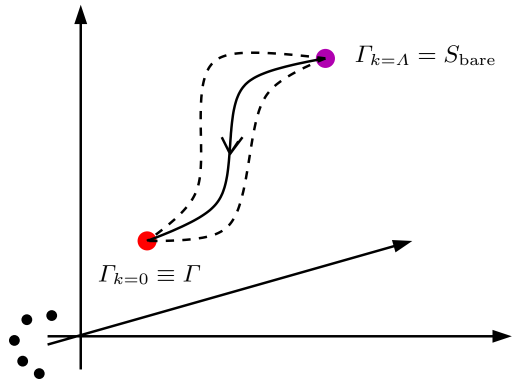


Fig. 3. Sketch of the RG flow in theory space. Each axis labels a different operator which spans the effective action, e.g., ϕ^2, $(\partial\phi)^2$, etc. Once the initial conditions in terms of the bare action $\Gamma_{k=\Lambda} = S_{\text{bare}}$ are given, the solution to the flow equation (28) is a trajectory (solid line) in this space of action functionals, ending at the full quantum effective action $\Gamma \equiv \Gamma_{k=0}$. A change of the regulator R_k can modify the trajectory (dashed lines), but the end point Γ stays always the same.

In practice, the versatility of the flow equation is its most important strength: beyond perturbation theory, various systematic approximation schemes exist which can be summarized under the *method of truncations*.

A first example for such an approximation scheme has already been given above in Eq. (9), the vertex expansion, which now reads

$$\Gamma_k[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^D x_1 \ldots \mathrm{d}^D x_n \, \Gamma_k^{(n)}(x_1, \ldots, x_n) \, \phi(x_1) \ldots \phi(x_n). \tag{29}$$

Upon inserting this expansion into the flow equation (28), we obtain flow equations for the vertex functions $\Gamma_k^{(n)}$ which interpolate between the bare and the fully dressed vertices. These equations are similar but not identical to Dyson-Schwinger equations, as will be discussed in more detail below.

As a second example, let us introduce the *operator expansion* which constructs the effective action from operators of increasing mass dimension. Focusing in particular on derivative operators, we arrive at the gradient expansion. For instance, for a theory with one real scalar field, we obtain

$$\Gamma_k = \int \mathrm{d}^D x \left[V_k(\phi) + \frac{1}{2} Z_k(\phi) \, (\partial_\mu \phi)^2 + \mathcal{O}(\partial^4) \right], \tag{30}$$

where, for instance, $V_k(\phi)$ corresponds to the effective potential.

Further examples of this type can easily be constructed by combining these two expansions in various ways. Formally, expansions of the effective action should be *systematic*; this implies that a classification scheme exists which classifies all possible building blocks of the effective action and relates them to a definite order in the expansion. Truncations based on such expansions should also be *consistent* in the sense that, once the maximal order of the truncated expansion is chosen, all terms up to this order are kept in the flow equation.

Systematics and consistency are, of course, only a necessary condition for the construction of a reliable truncation – they are not necessarily sufficient. As a word of caution, let us stress that these two conditions do not guarantee a rapid convergence of the truncated effective action towards the true result. As for any expansion, convergence properties have to be checked separately, in order to estimate or even control the truncation errors.

Only one general recipe for the construction of a truncation exists: let your truncation scheme be guided by physics by making sure that the truncation includes the most relevant degrees of freedom of a given problem.

2.3 Euclidean Anharmonic Oscillator

Let me illustrate the capabilities of the flow equation by a simple example: a 0+1 dimensional real scalar field theory, or, in other words, the Euclidean quantum mechanical anharmonic oscillator. This system has first been studied by RG techniques in [30]. The bare action that we want to quantize is

$$S = \int d\tau \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 + \frac{\lambda}{24} x^4 \right), \tag{31}$$

with $\omega^2, \lambda > 0$.[5] We will mainly be interested in the determination of the ground state energy which we expect to be predominantly influenced by the effective potential. Hence, we consider the truncation

$$\Gamma_k[x] = \int d\tau \left(\frac{1}{2}\dot{x}^2 + V_k(x) \right). \tag{32}$$

For a concrete computation, we have to choose a regulator which conforms to the conditions (13)-(15). The following choice is not only simple and convenient, it is also an *optimal* choice for the present problem, since it improves the stability properties of our flow equation [33],

$$R_k(p) = \left(k^2 - p^2\right) \theta(k^2 - p^2), \tag{33}$$

which implies that $\partial_t R_k = 2k^2\, \theta(k^2 - p^2)$. On the right-hand side of the flow equation (28), we need $\Gamma_k^{(2)} = (-\partial_\tau^2 + V_k''(x))\delta(\tau - \tau')$. In order to project the flow equation onto the flow of the effective potential, it suffices to consider the special case $x = \text{const.}$, for which the right-hand side can immediately be Fourier transformed, e.g., $-\partial_\tau^2 \to p_\tau^2$. We obtain the flow of the effective potential:

[5] Also the double-well potential with $\omega^2 < 0$ can be studied with RG techniques, see [31, 32].

$$\partial_t V_k(x) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dp_\tau}{2\pi}\, \frac{2k^2\theta(k^2-p_\tau^2)}{k^2+V_k''(x)}$$
$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}k}V_k(x) = \frac{1}{\pi}\,\frac{k^2}{k^2+V_k''(x)}. \tag{34}$$

This is a partial differential equation for the effective potential. Aiming at the ground state energy, it suffices to study a polynomial expansion of the potential,

$$V_k(x) = \frac{1}{2}\omega_k^2 x^2 + \frac{1}{24}\lambda_k x^4 + \ldots + \tilde{E}_k, \tag{35}$$

where the effects of the fluctuations are now encoded in a scale-dependent frequency ω_k and coupling λ_k; their initial conditions at $k = \Lambda$ are given by the parameters ω and λ in the bare action (31). Unfortunately, $\tilde{E}_k$ is not identical to the desired ground state energy $E_{0,k}$, but differs by further x-independent contributions. This becomes already clear by looking at the UV limit $k \to \Lambda$, where the regulator term becomes $\sim \frac{1}{2}\Lambda^2 x^2$, contributing a harmonic-oscillator-like ground state energy $\sim \frac{1}{2}\Lambda$ to $\tilde{E}$. In order to extract the true ground state energy from the flow of $\tilde{E}_k$,

$$\frac{\mathrm{d}}{\mathrm{d}k}\tilde{E}_k = \frac{1}{\pi}\,\frac{k^2}{k^2+\omega_k^2}, \tag{36}$$

we can perform a controlled subtraction to avoid the build-up of the unphysical contributions: in the limit $\lambda = \omega = 0$, the ground state energy has to stay zero, $E_{0,k} = 0$. This fixes the subtraction term for Eq. (36), which then reads:

$$\frac{\mathrm{d}}{\mathrm{d}k}E_{0,k} = \frac{1}{\pi}\left(\frac{k^2}{k^2+\omega_k^2} - 1\right), \tag{37}$$

Expanding Eq. (34) to higher orders, yields

$$\frac{\mathrm{d}}{\mathrm{d}k}\omega_k^2 = -\frac{2}{\pi}\,\frac{k^2}{(k^2+\omega_k^2)^2}\,\frac{\lambda_k}{2}, \tag{38}$$
$$\frac{\mathrm{d}}{\mathrm{d}k}\lambda_k = \frac{24}{\pi}\,\frac{k^2}{(k^2+\omega_k^2)^3}\,\left(\frac{\lambda_k}{2}\right)^2 + \ldots, \tag{39}$$

where the ellipsis in the last equation denotes contributions from higher-order terms $\sim x^6$, which we neglect here for simplicity. We have boiled the flow equation down to a coupled set of first-order ordinary differential equations, which can be viewed as the RG β functions of the generalized couplings $E_{0,k}, \omega_k, \lambda_k$. These equations can diagrammatically be displayed as in Fig. 4. The diagrams look very similar to one-loop perturbative diagrams, but there are important differences: all internal lines and vertices correspond to full propagators and full vertices (in our simple truncation here, the vertex represents a full running λ_k and the propagators contain the running ω_k^2). Furthermore, one propagator in each loop carries a regulator insertion, implying the replacement $G \to G_k \partial_t R_k G_k$ in comparison with perturbative diagrams.

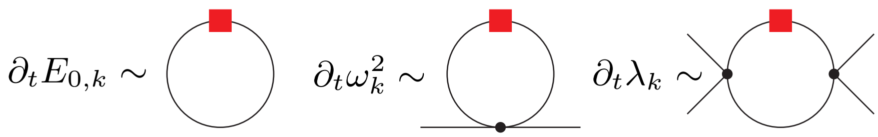


Fig. 4. Diagrammatic representation of Eqs. (37)-(39). The diagrams look similar to one-loop perturbative diagrams with all internal propagators and vertices being fully dressed quantities. One internal line always carries the regulator insertion $\partial_t R_k$ (filled box). (One further diagram for $\partial_t \lambda_k$ involving a 6-point vertex is dropped, as in Eq. (39).)

It is instructive, to solve these three equations (37)-(39) with various approximations. We begin with dropping the anharmonic coupling $\lambda = 0$, implying that $\omega_k = \omega_{k=\Lambda} \equiv \omega$; integrating the remaining flow of the ground state energy yields

$$E_0 \equiv E_{0,k=0} = \int_0^\infty \mathrm{d}k \, \frac{\mathrm{d}}{\mathrm{d}k} E_{0,k} \overset{(37)}{=} \frac{1}{2}\,\omega, \tag{40}$$

which corresponds to the ground state energy of the harmonic oscillator, as it should ($\hbar = 1$).

Now, let us try to do a bit better with the minimal nontrivial approximation: we drop the running of the anharmonic coupling $\lambda_k \to \lambda$, integrate the flow of the frequency (38), insert the resulting ω_k into Eq. (37) and integrate the energy. Expanding the result perturbatively in λ, we find

$$E_0 = \frac{1}{2}\omega + \frac{3}{4}\,\omega \left(\frac{\lambda}{24\omega^3}\right) - \frac{82}{40}\,\omega \left(\frac{\lambda}{24\omega^3}\right)^2 + \ldots, \tag{41}$$

which can immediately be compared with direct 2nd-order perturbation theory [34],

$$E_0^{\mathrm{PT}} = \frac{1}{2}\omega + \frac{3}{4}\,\omega \left(\frac{\lambda}{24\omega^3}\right) - \frac{105}{40}\,\omega \left(\frac{\lambda}{24\omega^3}\right)^2 + \ldots. \tag{42}$$

Our first-order "one-loop" result agrees exactly with perturbation theory, whereas the second-order "two-loop" coefficient has the right sign and order of magnitude but comes out too small with a $\sim 20\%$ error. However, it should be kept in mind that we have obtained this two-loop estimate from a cheap calculation which involved only a one-loop integral with an RG-improved propagator.

Of course, we do not have to stop with perturbation theory. We can look at the full integrated result based on Eqs. (37) and (38) for any value of λ. For instance, let us boldly study the strong-coupling limit, where the asymptotics is known to be of the form,

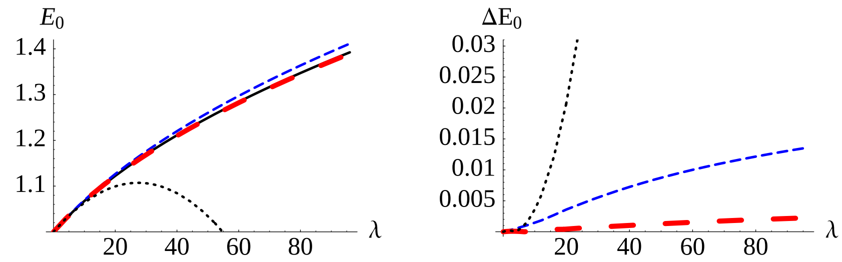


Fig. 5. Left panel: ground state energy of the anharmonic oscillator for $\omega = 2$ versus the anharmonic coupling λ: exact result (solid/black line), 2-loop perturbation theory (dotted/black line), flow-equation estimate based on Eqs. (37),(38) (short-dashed/blue line) or on Eqs. (37)-(39) (long-dashed/red line). Right panel: Relative error of the different estimates.

$$E_0 = \left(\frac{\lambda}{24}\right)^{1/3} \left[\alpha_0 + \mathcal{O}\left(\lambda^{-2/3}\right)\right]. \tag{43}$$

The constant α_0 has been determined in [35] to a high precision by means of large-order variational perturbation theory: $\alpha_0 = 0.66798\ldots$.

With the simplest nontrivial approximation based on Eqs. (37) and (38), we obtain $\alpha_0^{(37),(38)} = 0.6920\ldots$, which differs from the full solution by merely 4%. Now, solving all three equations (37)-(39) simultaneously, we find $\alpha_0^{(37)-(39)} = 0.6620\ldots$, which corresponds to a 1% error.

A plot of the ground state energy is depicted in Fig. 5 (left panel) for $\omega = 2$ and $\lambda = 0\ldots 100$. Whereas the perturbative estimate (dotted line) is a good approximation for small λ, it becomes useless for larger coupling. Already the simplest approximation based on Eqs. (37) and (38) (short-dashed/blue line) is a reasonable estimate of the exact result (solid/black line), obtained from the exact integration of the Schrödinger equation as quoted in [36]. The flow equation estimate based on Eqs. (37)-(39) (long-dashed/red line) is hardly distinguishable from the exact result. For better visibility, the relative errors of these estimates are shown in Fig. 5 (right panel). Over the whole range of λ, the error of our estimate based on Eqs. (37)-(39) (long-dashed/red line) does not exceed 0.3%.

The quality of the result is remarkable in view of the extremely simple approximation of the full flow. In particular, the truncation to a low-order polynomial potential does not seem to be justified at large coupling. In fact, there is no reason, why the dropped higher-order terms, e.g., $\sim x^6$, should be small compared to the terms kept.

The lesson to be learned is the following: it does not really matter whether the terms dropped are small compared to the terms kept. It only matters whether their influence on the terms that belong to the truncation is small or large.

2.4 Further reading: regulator dependence and optimization*

The reliability of the solution of a truncated RG flow is an important question that needs to be addressed in detail for each application. In absence of an obvious small expansion parameter, the convergence of any systematic and consistent expansion of the effective action can be checked by studying the quantitative influence of higher-order terms. Whereas such computations can become rather extensive, an immediate check can be performed by regulator studies. For the exact flow, physical observables evaluated at $k = 0$ are, of course, regulator independent by construction. However, truncations generically induce spurious regulator dependencies, the amount of which provides a measure for the importance of higher-order terms outside a given truncation. Resulting regulator dependencies of physical observables can thus be used for a quantitative error estimate in a rather direct manner.

Moreover, the freedom to choose the regulator function within the mild set of conditions provided by Eqs. (13)-(15) can also actively be used for an optimization of the flow. A truncated flow is optimized if the results for physical observables lie as close as possible to the true results; the influence of operators outside a given truncation scheme on the physical results is then minimized. If a truncated flow is optimized at each order in a systematic expansion, the physical results converge most rapidly towards the true result, also implying that the optimized flows exhibit enhanced numerical stability. Optimization of RG flows has conceptually been advanced in [37, 38, 33, 39, 8]. In particular, an optimization criterion for the IR regulated propagator at vanishing field based on stability considerations has lead to the construction of optimized regulators for general truncation schemes [33]; we have used such a regulator already in Eq. (33). A full functional approach to optimization has been presented in [8]. Loosely speaking, the optimal RG flow within a truncation is identified with the most direct, i.e., shortest trajectory in theory space.

Since optimization helps improving quantitative predictions, enhances numerical stability and can at the same time be used to reduce technical effort, it is of high practical relevance. Optimization is therefore recommended for all modern applications of the functional RG.

3 Functional RG for Gauge Theories

3.1 RG flow equations and symmetries

Before we embark on the complex machinery of quantum field theories with non-abelian gauge symmetries, let us study the interplay between flow equations and symmetries from a more general perspective with emphasis on the structural aspects.

Consider a QFT which is invariant under a continuous symmetry transformation which can be realized linearly on the fields; let $\mathcal{G}$ be the generator of an infinitesimal version of this transformation, i.e., $\mathcal{G}\phi$ is linear in ϕ. For example, a global $\mathrm{O}(N)$ symmetry in a QFT for N-component scalar fields is generated by

$$\mathcal{G}^a = -f^{abc} \int \mathrm{d}^D x\, \phi^b(x)\, \frac{\delta}{\delta\phi^c(x)}. \tag{44}$$

Incidentally, a local symmetry would be generated by the analogue of Eq. (44) without the spacetime integral. Together with the invariance of the measure under this symmetry, cf. Eq. (2), the invariance of the QFT can be stated by

$$0 = \frac{1}{Z} \int \mathcal{D}\varphi\, \mathcal{G}\, \mathrm{e}^{-S+\int J\varphi}. \tag{45}$$

In other words, a transformation of the action can be undone by a transformation of the measure.

What does this symmetry of the QFT imply for the effective action? First, we observe that Eq. (45) yields

$$\begin{aligned} 0 &= \frac{1}{Z} \int \mathcal{D}\varphi \left(-(\mathcal{G}S) + \int J(\mathcal{G}\varphi) \right) \mathrm{e}^{-S+\int J\varphi} \\ &= -\langle \mathcal{G}S \rangle_J + \mathrm{e}^{-W[J]} \int J(\mathcal{G}\varphi)|_{\varphi=\frac{\delta}{\delta J}} \mathrm{e}^{W[J]}. \end{aligned} \tag{46}$$

Performing the Legendre transform, we find at $J = J_{\mathrm{sup}} \equiv J[\phi]$:

$$0 = -\langle \mathcal{G}S \rangle_{J[\phi]} + \int \frac{\delta\Gamma}{\delta\phi} \mathcal{G}\phi. \tag{47}$$

The last term is nothing but $\mathcal{G}\Gamma$, and we obtain the important identity (Ward identity)

$$\mathcal{G}\Gamma[\phi] = \langle \mathcal{G}S \rangle_{J[\phi]}. \tag{48}$$

It demonstrates that the effective action is invariant under a symmetry if the bare action as well as the measure are invariant.

On this level, the statement sounds rather trivial, but it can readily be generalized to the effective average action by keeping track of the regulator term,

$$\mathcal{G}\Gamma_k[\phi] = \langle \mathcal{G}(S + \Delta S_k) \rangle_{J[\phi]} - \mathcal{G}\Delta S_k[\phi]. \tag{49}$$

Here, we learn that the whole RG trajectory Γ_k is invariant under the symmetry if $\mathcal{G}S = 0$ and $\mathcal{G}\Delta S_k = 0$. This is the case if the regulator preserves the symmetry. For instance for the globally $\mathrm{O}(N)$-symmetric theory, a regulator of the form

$$\Delta S_k = \frac{1}{2} \int \frac{\mathrm{d}^D p}{(2\pi)^D}\, \varphi^a(-p) \delta^{ab} R_k(p) \varphi^b(p) \tag{50}$$

is such an invariant regulator.

Note the following crucial point: since the regulator vanishes, $\Delta S_k \to 0$, for $k \to 0$, Eq. (49) appears to tell us that $\mathcal{G}\Gamma_{k=0}[\phi] = \langle \mathcal{G}S \rangle_{J[\phi]}$ always

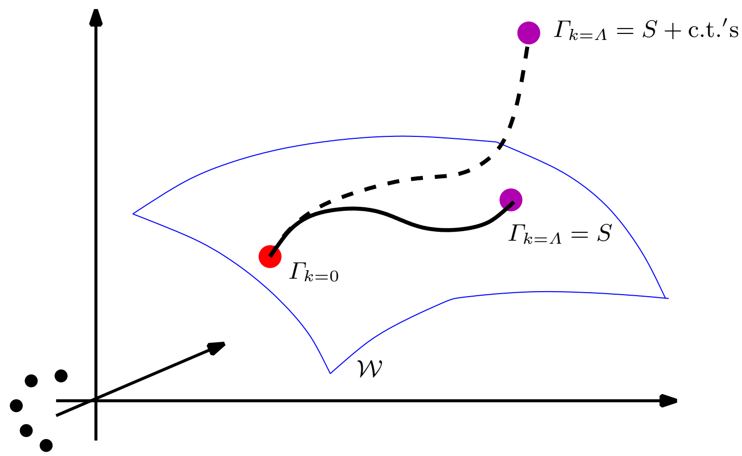


Fig. 6. Sketch of the RG flow in theory space with symmetries: the symmetry relation cuts out a hypersurface $\mathcal{W}$ where the action is invariant under the given symmetry. A symmetric regulator keeps the invariance explicitly, such that the RG trajectory always stays inside $\mathcal{W}$ (solid line). A non-symmetric regulator can still be used if non-symmetric counterterms (c.t.'s) are chosen such that they eat up the non-symmetric flow contributions (dashed line) by virtue of the Ward identity Eq. (51). Also in this case, the resulting effective action is invariant, $\Gamma_{k=0} \in \mathcal{W}$.

holds, implying that the symmetry is always restored at $k = 0$ even for a non-symmetric regulator. This is indeed true, but requires that the initial conditions at $k = \Lambda$ have to be carefully chosen in a highly non-symmetric manner, since

$$\mathcal{G}\Gamma_\Lambda[\phi] = \langle \mathcal{G}S \rangle + \langle \mathcal{G}\Delta S_{k=\Lambda} \rangle - \mathcal{G}\Delta S_{k=\Lambda}. \tag{51}$$

Here, the last two terms do generally not cancel each other. Therefore, even for non-symmetric regulators, the Ward identity tells us how to choose initial conditions with non-symmetric UV counterterms, such that the latter are exactly eaten up by non-symmetric flow contributions, see Fig. 6.

In general, it is advisable to use a symmetric regulator, since the space of symmetric actions is smaller, implying that fewer couplings have to be studied in a truncation. However, if a symmetric regulator is not available, the flow equation together with the Ward identity can still be used. In fact, in gauge theories there is no simple formalism with a symmetric regulator.

3.2 Basics of Gauge Theories

Let us review a few basic elements of quantum gauge theories for reasons of completeness and in order to introduce our notation.

Consider the classical Yang-Mills action,

$$S_{\mathrm{YM}} = \int \mathrm{d}^D x \, \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}, \tag{52}$$

with the field strength

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \tag{53}$$

where the gauge field A^a_μ carries an internal-symmetry index (color), and f^{abc} are the structure constants of a compact non-abelian Lie group. The hermitean generators T^a of this group form a Lie algebra and satisfy

$$[T^a, T^b] = \mathrm{i} f^{abc} T^c, \tag{54}$$

e.g., $a, b, c, = 1, 2, \dots, N_\mathrm{c}^2 - 1$ for the group $\mathrm{SU}(N_\mathrm{c})$. (In the fundamental representation, the T^a are hermitean $N_\mathrm{c} \times N_\mathrm{c}$ matrices which can be normalized to $\mathrm{tr}[T^a, T^b] = \frac{1}{2}\delta^{ab}$.)

The naive attempt to define the corresponding quantum gauge theory by

$$Z[J] \stackrel{?}{=} \int \mathcal{D}A \, \mathrm{e}^{-S_{\mathrm{YM}}[A] + \int J^a_\mu A^a_\mu} \tag{55}$$

fails and generically leads to ill-defined quantities plagued by infinities. The reason is that the measure $\mathcal{D}A^a_\mu$ contains a huge redundancy, since many gauge-field configurations A^a_μ are physically equivalent. Namely, the action is invariant under the local symmetry

$$A^a_\mu \to A^a_\mu - \partial_\mu \omega^a + g f^{abc} \omega^b A^c_\mu \equiv A^a_\mu + \delta A^a_\mu, \tag{56}$$

where $\omega^a(x)$ is considered to be infinitesimal and differentiable, but otherwise arbitrary. The set of all possible transformations forms the corresponding gauge group. The generator of this symmetry is

$$\mathcal{G}^a_A(x) = D^{ab}_\mu \frac{\delta}{\delta A^b_\mu}, \tag{57}$$

where $D^{ab}_\mu = \partial_\mu \delta^{ab} + g f^{abc} A^c_\mu$ denotes the covariant derivative in adjoint representation. It is a simple exercise to show that $\int \mathrm{d}^D x \, \omega^b \mathcal{G}^b_A A^a_\mu = \delta A^a_\mu$. The full symmetry transformation for non-infinitesimal $\omega^a(x)$ can be written as ($A_\mu \equiv A^a_\mu T^a$)

$$A_\mu \to A^\omega_\mu = U A_\mu U^{-1} - \frac{\mathrm{i}}{g} (\partial_\mu U) U^{-1}, \tag{58}$$

with $U = U[\omega] = \mathrm{e}^{-\mathrm{i} g \omega^a T^a}$ being an element of the Lie group. Field configurations which are connected by Eq. (58) are *gauge equivalent* and form the *gauge orbit*:

$$[A^{\mathrm{orbit}}_\mu] = \{ A^\omega_\mu \,|\, A_\mu = A^{\mathrm{ref}}_\mu, \quad U[\omega] \in \mathrm{SU}(N_\mathrm{c}) \}. \tag{59}$$

Here, A^{ref}_μ is a reference gauge field which is representative for the orbit.

In order to define the quantum theory, we would like to dispose of a measure which picks one representative gauge-field configuration out of each orbit. This is intended by choosing a gauge-fixing condition,

$$\mathsf{F}^a[A] = 0. \tag{60}$$

For instance, $\mathsf{F}^a = \partial_\mu A^a_\mu$ is an example for a Lorentz covariant gauge-fixing condition. Ideally, Eq. (60) should be satisfied by only one A^a_μ of each orbit. (As discussed below, this is actually impossible for standard smooth gauge-fixing conditions, owing to topological obstructions [40]).

Gauge fixing can be implemented in the generating functional by means of the Faddeev-Popov trick which is usually derived from

$$1 = \int \mathcal{D}\mathsf{F}^a\, \delta[\mathsf{F}^a] = \int \mathrm{d}\mu(\omega)\, \delta[\mathsf{F}^a(A^\omega)]\, \det\left(\frac{\delta \mathsf{F}^a[A^\omega]}{\delta\omega^b}\right), \tag{61}$$

where $\mathrm{d}\mu$ denotes the invariant Haar measure for an integration over the gauge group manifold (at each spacetime point). This rule is reminiscent to the corresponding rule for variable substitution in an ordinary integral, $1 = \int \mathrm{d}f\, \delta(f) = \int \mathrm{d}x\, \delta(f(x)) \left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|$, for $f(x)$ having only one zero. But as already this simple comparison shows, Eq. (61) is accompanied by the tacit assumption that $\mathsf{F}^a = 0$ picks only one representative and that the Faddeev-Popov determinant $\det(\delta\mathsf{F}/\delta\omega) > 0$. Both assumptions are generally not true; and both these properties, namely, that several gauge copies on the same gauge orbit all satisfy a given standard gauge-fixing condition and that the Faddeev-Popov determinant is not positive definite, are characteristic of the famous Gribov problem [41]. A direct solution to this problem in the functional integral formalism is by no means simple; nevertheless, let us assume here that a solution exists which renders the gauge-fixed functional integral well defined such that we can proceed with deriving the flow equation. We will return to this problem later in the discussion of the flow equation.

As discussed in standard textbooks, it is now possible to show that the Faddeev-Popov determinant is gauge invariant, $\varDelta_{\mathrm{FP}}[A^\omega] \equiv \det \frac{\delta\mathsf{F}^a[A^\omega]}{\delta\omega^b} = \varDelta_{\mathrm{FP}}[A]$. As a consequence, $\delta[\mathsf{F}^a[A]]\varDelta_{\mathrm{FP}}[A]$ can be inserted into the functional integral, such that the redundancy introduced by gauge symmetry is removed at least for perturbative amplitudes; this renders the perturbative amplitudes well defined and S matrix elements are, in fact, independent of the gauge-fixing condition. As a result, the Euclidean gauge-fixed generating functional, replacing the naive attempt Eq. (55), becomes

$$Z[J] = \mathrm{e}^{W[J]} = \int \mathcal{D}A\, \varDelta_{\mathrm{FP}}[A]\, \delta[\mathsf{F}^a[A]]\, \mathrm{e}^{-S_{\mathrm{YM}} + \int JA}. \tag{62}$$

The additional terms can be brought into the exponent:

$$\delta[\mathsf{F}^a[A]] \to \mathrm{e}^{-\frac{1}{2\alpha}\int \mathrm{d}^D x \mathsf{F}^a\mathsf{F}^a}\Big|_{\alpha\to 0} \equiv \mathrm{e}^{-S_{\mathrm{gf}}[A]}, \tag{63}$$

where we have used a Gaußian representation of the δ functional. The exponentiation of the Faddeev-Popov determinant can be done with Grassmann-valued anti-commuting real *ghost* fields $c, \bar{c}$, yielding

$$\Delta_{\rm FP}[A] = \int \mathcal{D}\bar{c}\mathcal{D}c\, \mathrm{e}^{-\int \mathrm{d}^D x\, \bar{c}^a \frac{\delta F^a}{\delta \omega^b} c^b} \equiv \int \mathcal{D}\bar{c}\mathcal{D}c\, \mathrm{e}^{-S_{\rm gh}}. \tag{64}$$

The ghost fields transform homogeneously,

$$c^a \to c^a + g f^{abc}\omega^b c^c, \quad \bar{c}^a \to \bar{c}^a + g f^{abc}\omega^b \bar{c}^c, \tag{65}$$

as induced by a corresponding generator,

$$\mathcal{G}^a_{\rm gh} = -g f^{abc}\left(c^c \frac{\delta}{\delta c^b} + \bar{c}^c \frac{\delta}{\delta \bar{c}^b}\right). \tag{66}$$

In perturbative S matrix elements, the ghosts can be shown to cancel unphysical redundant gauge degrees of freedom.

Now, we generalize the generating functional by coupling sources also to the ghosts, in order to treat them on the same footing as the gauge field,

$$Z[J,\eta,\bar{\eta}] = \mathrm{e}^{W[J,\eta,\bar{\eta}]} = \int \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\, \mathrm{e}^{-S_{\rm YM} - S_{\rm gh} - S_{\rm gf} + \int JA + \int \bar{\eta} c - \int \bar{c}\eta}. \tag{67}$$

The construction of the effective action Γ now proceeds in a standard fashion,

$$\Gamma[A,\bar{c},c] = \sup_{J,\eta,\bar{\eta}} \left(\int JA + \int \bar{\eta} c - \int \bar{c}\eta - W[J,\eta,\bar{\eta}]\right). \tag{68}$$

Since Γ is the result of a gauge-fixed construction, it is not manifestly gauge invariant. Gauge invariance is now encoded in a constraint given by the Ward identity Eq. (48) applied to the present case with the generator

$$\mathcal{G}^a(x) = \mathcal{G}^a_A(x) + \mathcal{G}^a_{\rm gh}(x) = D^{ab}_\mu \frac{\delta}{\delta A^b_\mu} - g f^{abc}\left(c^c \frac{\delta}{\delta c^b} + \bar{c}^c \frac{\delta}{\delta \bar{c}^b}\right). \tag{69}$$

Since $\mathcal{G}^a S_{\rm YM} = 0$, the Ward identity boils down to

$$\mathcal{W} := \mathcal{G}^a \Gamma[A,\bar{c},c] - \langle \mathcal{G}^a (S_{\rm gf} + S_{\rm gh})\rangle = 0, \tag{70}$$

which in the context of nonabelian gauge theories is also commonly referred to as Ward-Takahashi identity (WTI). For instance in the Landau gauge,

$$\mathsf{F}^a[A] = \partial_\mu A^a_\mu, \quad \frac{\delta \mathsf{F}^a[A^\omega]}{\delta \omega^b} = -\partial_\mu D^{ab}_\mu[A], \tag{71}$$

together with the gauge parameter $\alpha \to 0$, cf. Eq. (63), we can work out this identity more explicitly by computing $\mathcal{G}^a S_{\rm gf,gh}$, e.g., in momentum space. This is done in Subsect. 3.4. As an example for how the Ward-Takahashi identity constrains the effective action, let us consider a gluon mass term, $\Gamma_{\rm mass} = \frac{1}{2}\int m_A^2 A^a_\mu A^a_\mu$. It can be shown order by order in perturbation theory that the Ward-Takahashi identity enforces this gluon mass to vanish, $m_A^2 = 0$; for more details, see Subsect. 3.4. Hence, the gluon is protected against acquiring a mass by perturbative quantum fluctuations because of gauge invariance.

3.3 RG flow equation for gauge theories

From the gauge-fixed generating functional, the RG flow equation for Γ_k can straightforwardly be derived along the lines of Subsect. 2.2. Using a regulator term,

$$\begin{aligned}\Delta S_k &= \frac{1}{2}\int \frac{\mathrm{d}^D p}{(2\pi)^D}\, A^a_\mu(-p)\, (R_{k,A})^{ab}_{\mu\nu}(p)\, A^b_\nu(p) \\ &= + \int \frac{\mathrm{d}^D p}{(2\pi)^D}\, \bar{c}^a(p)\, (R_{k,\mathrm{gh}})^{ab}(p)\, c^b(p),\end{aligned} \tag{72}$$

we obtain the flow equation

$$\begin{aligned}\partial_t \Gamma_k[A,\bar{c},c] &= \frac{1}{2}\,\mathrm{Tr}\,\partial_t R_{k,A}[(\Gamma_k^{(2)}+R_k)^{-1}]_A - \mathrm{Tr}\partial_t R_{k,\mathrm{gh}}[(\Gamma_k^{(2)}+R_k)^{-1}]_{\mathrm{gh}} \\ &\equiv \frac{1}{2}\,\mathrm{STr}\,\partial_t R_k(\Gamma_k^{(2)}+R_k)^{-1}.\end{aligned} \tag{73}$$

The minus sign in front of the ghost term arises because of the anti-commuting nature of these Grassmann-valued fields; the super-trace in the second line of Eq. (73) takes this sign into account. In our notation in Eq. (73), $\Gamma_k^{(2)}$ is also matrix-valued in field space, i.e., with respect to $(A,\bar{c},c)$; therefore, $[(\Gamma^{(2)}+R_k)^{-1}]_A$ denotes the gluon component of the full inverse of $\Gamma^{(2)}+R_k$ (and not just the inverse of $\delta^2\Gamma_k/\delta A\delta A + R_{k,A}$).

Is this a gauge-invariant flow? Manifest gauge invariance is certainly lost, because the regulator is not gauge invariant; e.g., at small p, the regulator – here being similar to a mass term – is forbidden by gauge invariance, as discussed above.[6] But manifest gauge invariance is anyway lost, owing to the gauge-fixing procedure. Gauge symmetry is encoded in the Ward-Takahashi identity. From this viewpoint, the regulator is merely another source of explicit gauge-symmetry breaking, giving rise to further terms in the Ward-Takahashi identity, the form of which we can directly read off from Eq. (49):

$$\mathcal{W}_k := \mathcal{G}\Gamma_k + \mathcal{G}\Delta S_k - \langle \mathcal{G}(S_{\mathrm{gf}} + S_{\mathrm{gh}} + \Delta S_k)\rangle = 0. \tag{74}$$

Owing to the additional regulator terms, this equation is called *modified Ward-Takahashi identity* (mWTI). With ΔS_k being quadratic in the field variables, these regulator-dependent terms have a one-loop structure, since $\langle \mathcal{G}\Delta S_k\rangle - \mathcal{G}\Delta S_k$ corresponds to an integral over the connected 2-point function with a

[6] One may wonder whether a gauge-invariant flow can be set up with a gauge-invariant regularization procedure. In fact, this is an active line of research, and various promising formalisms have been developed so far [42, 43, 44]. However, the price to be paid for the resulting simple gauge constraints comes in the form of nontrivial Nielsen identities, non-localities or extensive algebraic constructions. For practical application, we thus consider the standard formulation described here as the most efficient approach so far.

regulator insertion $\sim R_k$. Since the standard Ward-Takahashi identity already involves loop terms (cf. Subsect. 3.4), the solution to Eq. (74) is no more difficult to find than that of the standard WTI.

As before, we observe that

$$\lim_{k\to 0} \mathcal{W}_k \equiv \mathcal{W}, \tag{75}$$

such that a solution to the mWTI $\mathcal{W}_k = 0$ satisfies the standard WTI, $\mathcal{W} = 0$, if the regulator is removed at $k = 0$. Such a solution is thus gauge invariant. Loosely speaking, the mWTI $\mathcal{W}_k = 0$ defines a modified gauge invariance that reduces to the physical gauge invariance for $k \to 0$. For a discussion of these modified symmetry constraints from different perspectives, see e.g., [19, 20, 21, 22, 45, 46, 47, 48, 8], or the review [3].

One further important observation is that the flow of the mWTI satisfies

$$\partial_t \mathcal{W}_k = -\frac{1}{2}\, G_k^{AB} \partial_t R_k^{AC} G_k^{CD}\, \frac{\delta}{\delta \Phi^B} \frac{\delta}{\delta \Phi^D}\, \mathcal{W}_k. \tag{76}$$

Here, we have used the collective field variable $\Phi = (A, \bar{c}, c)$. The collective indices $A, B, C, \dots$ label these components, and denote all discrete indices (color, Lorentz, etc.) as well as momenta; e.g., the flow equation reads in this notation: $\partial_t \Gamma_k = \frac{1}{2} \partial_t R_k^{AB} G_k^{BA}$. The derivation of Eq. (76) from Eq. (74) is indeed straightforward and a worthwhile exercise.

Let us draw an important conclusion based on Eq. (76): if we manage to find an effective action Γ_k which solves the mWTI $\mathcal{W}_k = 0$ at some scale k, then also the flow of the mWTI vanishes, $\partial_t \mathcal{W}_k = 0$. In other words, the mWTI is a fixed point under the RG flow. Now, if this Γ_k is connected with $\Gamma_{k'}$ at another scale k' by the flow equation, also $\Gamma_{k'}$ satisfies the mWTI at this new scale, $\mathcal{W}_{k'} = 0$. Gauge invariance at some scale therefore implies gauge invariance at all other scales, if the corresponding Γ_k's solve the flow equation. The whole concept of the mWTI is sketched and summarized in Fig. 7.

Unfortunately, the picture is not as rosy as it seems for a simple practical reason: in the general case, we will not be able to solve the flow equation exactly. Hence, the identity (76) and thus $\partial_t \mathcal{W}_k = 0$ will be violated on the same level of accuracy. This problem is severe if $\partial_t \mathcal{W}_k = 0$ is violated by RG-relevant operators, see Fig. 8; the latter are forbidden in the perturbative gauge-invariant theory.

In the perturbative domain where naive power-counting holds, an RG relevant operator is potentially given by the gluon mass term, $\frac{1}{2} \int m_A^2 A_\mu^a A_\mu^a$. Gauge invariance in the form of the standard WTI enforces $m_A^2 = 0$ as a consequence of $\mathcal{W} = 0$, as mentioned above. By contrast, such a bosonic mass term in a system without gauge symmetry would receive large contributions from fluctuations; perturbative diagrams are typically quadratically divergent in such systems. In a naively truncated RG flow of a gauge system, we can therefore expect the gluon mass to become large if the gauge symmetry is not

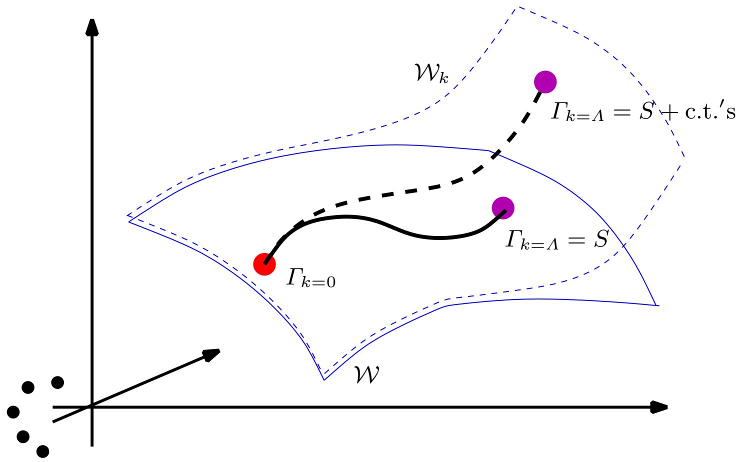


Fig. 7. Sketch of an RG flow with gauge symmetry in theory space: the standard Ward identity $\mathcal{W}$ cuts out a hypersurface of gauge-invariant action functionals. A gauge-invariant trajectory (solid line) would lie completely within this hypersurface. Instead, the presence of the regulator leads to the mWTI $\mathcal{W}_k$, cutting out a different hypersurface, which approaches $\mathcal{W}$ in the limit $k \to 0$. A solution to the flow equation stays within the $\mathcal{W}_k$ hypersurface (dashed line), implying gauge invariance of the final full action $\Gamma \equiv \Gamma_{k=0}$.

respected, $m_A^2 \sim g^2\Lambda^2$. Now, the mWTI assists to control the situation: since gauge symmetry is not manifestly present in our RG flow, we cannot expect the gluon mass to vanish at all values of k. As discussed in more detail in Subsect. 3.4, the gluon mass becomes of order $m_A^2 \sim g^2k^2$, as can be determined from $\mathcal{W}_k = 0$. As long as perturbative power-counting holds, this implies that $m_A^2 \to 0$ for $k \to 0$, and the gauge constraint becomes satisfied in the limit when the regulator is removed.

This consideration demonstrates that the mWTI can turn a potentially dangerous relevant operator, which may appear in some truncation, into an irrelevant harmless operator which dies out in the limit of vanishing regulator. From another viewpoint, the mWTI tells us precisely the right amount of gauge-symmetry breaking that we have to put at the UV scale Λ in the form of counterterms, such that this explicit breaking is ultimately eaten up by the fluctuation-induced breaking terms from the regulator, ending up with a perfectly gauge-invariant effective action.

This simple gluon-mass example teaches an important lesson: for the construction of a truncated gauge-invariant flow, the flow equation and the mWTI should be solved simultaneously within the truncation. As a standard recipe [49, 50], the flow equation can first be used to determine the flow of all *independent* operators (e.g., transverse propagators and transverse vertex projections) which are not constrained by $\mathcal{W}_k = 0$. Then use the mWTI to compute the remaining dependent operators (e.g., gluon mass, longitudinal terms). In

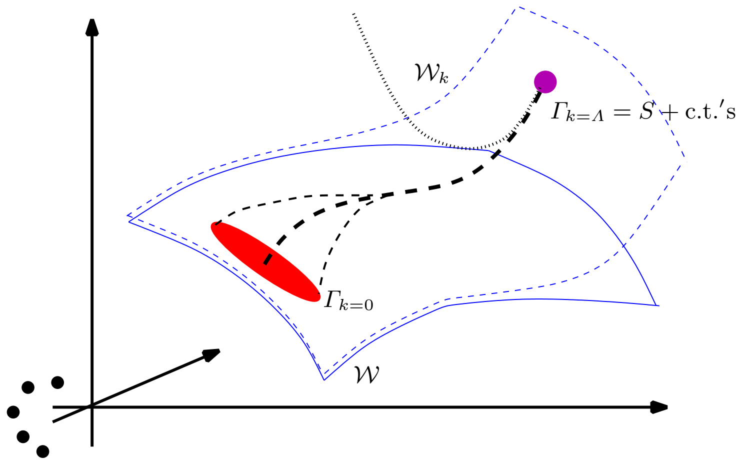


Fig. 8. Sketch of an RG flow with gauge symmetry in theory space: a truncation of the effective action introduces an error, implying a possible range of estimates for $\Gamma_{k=0}$ as depicted by the extended ellipse. The error can also have a component orthogonal to the $\mathcal{W}$ or $\mathcal{W}_k$ hypersurface, if the Ward identities are solved on the same level of accuracy as the truncated flow. A violation of the mWTI by RG relevant operators is particularly dangerous, since it can quickly drive the system away from the physical solution (dotted line) and thus must be avoided.

this manner, the gauge constraint is explicitly solved on the truncation and gauge-invariance of the truncation is guaranteed. [7] It turns out that the use of the mWTI (instead of the flow equation itself) for the computation of a dependent operator generically corresponds to a resummation of a larger class of diagrams [51].

For the remainder of this subsection, let us return to the Gribov problem discussed below Eq. (61). The fact that standard gauge fixings do not uniquely pick exactly one representative of each gauge orbit and that the Faddeev-Popov determinant hence is not positive makes the nonperturbative definition of the functional integral problematic. Any nonperturbative method which is related to the functional integral such as the flow equation therefore appears to face the same problem. As an example, let us concentrate on the Landau gauge and consider the set of all gauge-field configurations that satisfy the gauge-fixing condition $\mathsf{F}^a = \partial_\mu A_\mu^a = 0$; the Faddeev-Popov operator then is $\delta \mathsf{F}^a[A^\omega]/\delta\omega^b = -\partial_\mu D_\mu^{ab}[A]$. A perturbative expansion around $g \to 0$ goes along with the Faddeev-Popov operator at the origin of configuration space, $-\partial_\mu D_\mu^{ab}[A] \to -\partial^2$; since this Laplacian is a positive operator, the Gribov problem does not play a role in perturbation theory to any finite order.

[7] An alternative option could be to use only the flow equation together with a regulator that does automatically suppress artificial relevant operators. In fact, this is conceivable in the framework of optimization [8].

Moving away from the origin, it is useful to consider the following gauge-fixing functional, corresponding to the L_2 norm of the gauge potential along the gauge orbit,

$$\mathcal{F}_A[\omega] \equiv ||A^\omega||^2 = ||A||^2 + 2\int_x \omega^a \partial_\mu A_\mu^a + \int_x \omega^a(-\partial_\mu D_\mu^{ab})\omega^b + \mathcal{O}(\omega^3), \quad (77)$$

where we have expanded A_μ^ω of Eq. (58) to second order in ω. We can identify transverse gauge potentials that satisfy the gauge condition as the stationary points of $\mathcal{F}_A[\omega]$. The Gribov problem implies that a gauge orbit does not just contain one but many stationary points of $\mathcal{F}_A[\omega]$. We observe that the subset of stationary points given by (local) minima of $\mathcal{F}_A[\omega]$ corresponds to a positive Faddeev-Popov operator $(-\partial_\mu D_\mu^{ab}) > 0$; this subset constitutes the *Gribov region* Ω_{G}. Restricting the gauge-field integration to the Gribov region, $\int \mathcal{D}A \to \int_{\Omega_{\mathrm{G}}} \mathcal{D}A$, cures the most pressing problem of having a potentially ill-defined generating functional owing to a non-positive Faddeev-Popov determinant. Let us list some important properties of the Gribov region, as detailed in [52]: (i) Ω_{G} contains the origin of configuration space and thus all perturbatively relevant field configurations; (ii) the Gribov region is convex and bounded by the *(first) Gribov horizon* $\partial\Omega_{\mathrm{G}}$, consisting of those field configurations for which the lowest eigenvalue of the Faddeev-Popov operator vanishes; hence $\Delta_{\mathrm{FP}} = 0$ on $\partial\Omega_{\mathrm{G}}$.

Does this restriction of the gauge-field integration to the Gribov region modify the flow equation? In order to answer this question, let us go back to the functional-integral equation for the effective action (without IR regulator) in Eq. (8),

$$\mathrm{e}^{-\Gamma[\phi]} = \int \mathcal{D}\varphi \, \exp\left(-S[\phi+\varphi] + \int \frac{\delta\Gamma[\phi]}{\delta\phi}\,\varphi\right), \quad (78)$$

with the supplementary condition that $\langle\varphi\rangle = 0$, owing to the shifted integration variable, cf. Eq. (8). Differentiating both sides with respect to ϕ yields

$$\frac{\delta\Gamma[\phi]}{\delta\phi} = \left\langle \frac{\delta S[\varphi+\phi]}{\delta\phi} \right\rangle_{J[\phi]}, \quad (79)$$

which is a compact representation of the Dyson-Schwinger equations. Note that the same equation can be obtained from the following identity:

$$0 = \int \mathcal{D}\varphi \, \frac{\delta}{\delta\varphi} \mathrm{e}^{-S[\phi+\varphi] + \int \frac{\delta\Gamma[\phi]}{\delta\phi}\varphi}, \quad (80)$$

which holds, since the integrand is a total derivative. No boundary terms appear here, because the action typically goes to infinity, $S \to \infty$, for an unconstrained field $\varphi \to \infty$.

Now, the crucial point for a quantum gauge theory with Faddeev-Popov gauge fixing is that the identity corresponding to Eq. (80) holds also for the constrained integration domain Ω_{G}. No boundary term arises, simply because the Faddeev-Popov operator and thus the integrand vanishes on the boundary $\partial\Omega_{\mathrm{G}}$. We conclude that the Dyson-Schwinger equations are not modified by the restriction to the Gribov region. Finally, the same argument can be transfered to the flow-equation formalism by noting that the effective average action has to satisfy an identity similar to Eq. (78) including the regulator,

$$\mathrm{e}^{-\Gamma_k[\phi]-\Delta S_k[\phi]} = \int \mathcal{D}\varphi \, \exp\left(-S[\phi+\varphi] - \Delta S_k[\phi+\varphi] + \int \frac{\delta(\Gamma_k[\phi]+\Delta S[\phi])}{\delta\phi}\,\varphi\right), \tag{81}$$

with corresponding IR regulated Dyson-Schwinger equations,

$$\frac{\delta(\Gamma_k[\phi]+\Delta S_k[\phi])}{\delta\phi} = \left\langle \frac{\delta(S[\varphi+\phi]+\Delta S_k[\varphi+\phi])}{\delta\phi} \right\rangle_{J[\phi]}. \tag{82}$$

The latter can again be obtained from a functional integral over a total derivative similar to Eq. (80), indicating that a restriction to the Gribov region does not modify Eq. (82). The final step of the argument consists in noting that the scale derivative ∂_t of Eq. (82) yields the flow equation (once differentiated with respect to ϕ).

To summarize, solving quantum gauge theories by the construction of correlation functions by means of functional methods (Dyson-Schwinger equations or flow equations) precisely corresponds to an approach with a build-in restriction to the Gribov region as an attempt to solve the Gribov problem. From a flow-equation perspective, the argument can even be turned around: taking the viewpoint that the quantum gauge theory is defined by the flow equation, we can initiate the flow in the perturbative deep UV where the Faddeev-Popov determinant is guaranteed to be positive. Solving the flow, a resulting stable trajectory necessarily stays within the Gribov region.

Let us close this section with the remark that the picture developed so far is not yet complete. The restriction to the Gribov region only removes the problem of the non-positive Faddeev-Popov determinant. It does not guarantee that we have integrated over the configuration space by picking only one representative of each gauge orbit. In fact, even within the Gribov region, there are still Gribov copies. Therefore, the integration domain in gauge configuration space has to be restricted even further by picking the global minimum of $\mathcal{F}_A[\omega]$ in Eq. (77). The resulting space is known as the *fundamental modular region* Λ. In practice, the explicit construction of Λ is difficult; for instance, finding the global minimum of the gauge-fixing functional $\mathcal{F}_A[\omega]$ on the lattice, corresponds to an extremely involved spin-glass problem. However, it has been argued in [52] within a stochastic-quantization approach that the problem of Gribov copies within the Gribov region does not affect the correlation functions and their computation. This stresses even further the potential of functional methods for nonperturbative problems in gauge theories.

3.4 Ward-Takahashi identity*

The following subsection is devoted to a detailed discussion of the gauge constraints in the form of the Ward-Takahashi identity (WTI) and its modified counterpart in the presence of the regulator (mWTI). Let us start with the standard WTI which we derived already in a compact notation in Eq. (70),

$$\mathcal{W} := \mathcal{G}^a \Gamma[A, \bar{c}, c] - \langle \mathcal{G}^a (S_{\rm gf} + S_{\rm gh}) \rangle = 0, \tag{83}$$

which represents the gauge-symmetry encoding constraint that the effective action Γ has to satisfy in a gauge-fixed formulation. In order to work out the single terms more explicitly, it is useful to go to momentum space; we use the Fourier conventions

$$A_\mu^a(x) = \int_q \mathrm{e}^{\mathrm{i}qx} A_\mu^a(q), \quad c^a(x) = \int_q \mathrm{e}^{\mathrm{i}qx} c^a(q), \quad \bar{c}^a(x) = \int_q \mathrm{e}^{-\mathrm{i}qx} \bar{c}^a(q), \tag{84}$$

where $\int_q \equiv \int \frac{\mathrm{d}^D q}{(2\pi)^D}$. This implies for the functional derivatives, for instance,

$$\frac{\delta}{\delta A_\mu^a(x)} = \int_q \mathrm{e}^{-\mathrm{i}qx} \frac{\delta}{\delta A_\mu^a(q)}, \quad \text{etc.} \tag{85}$$

As a result, the generator of gauge transformations $\mathcal{G}^a$ reads in momentum space

$$\begin{aligned}\mathcal{G}^a(p) = {} & \mathrm{i}p_\mu \frac{\delta}{\delta A_\mu^a(-p)} \\ & - g f^{abc} \int_q \left[A_\mu^c(q) \frac{\delta}{\delta A_\mu^b(q-p)} + c^c(q) \frac{\delta}{\delta c^b(q-p)} + \bar{c}^c(q) \frac{\delta}{\delta \bar{c}^b(q-p)} \right].\end{aligned} \tag{86}$$

This allows us to compute the building blocks of the last two terms in Eq. (83) in momentum space; we obtain the gauge transforms

$$\mathcal{G}^a(p) S_{\rm gf} = \frac{\mathrm{i}}{\alpha} p^2 p_\mu A_\mu^a(p) - \frac{1}{\alpha} g f^{abc} \int_q A_\mu^c(q) (p-q)_\mu (p-q)_\nu A_\nu^b(p-q), \tag{87}$$

$$\begin{aligned}\mathcal{G}^a(p) S_{\rm gh} = {} & - g f^{abc} \int_q \bar{c}^c(q) p \cdot (q+p) c^b(q+p) \\ & - \mathrm{i} g^2 f^{feb} f^{abc} \int_{q_1, q_2} p_\mu \bar{c}^c(q_1) A_\mu^e(p+q_1+q_2) c^f(q_2).\end{aligned} \tag{88}$$

Upon insertion into Eq. (83), we arrive at a explicit representation of the WTI in terms of full correlation functions in the presence of a source J which is field dependent by virtue of the Legendre transform, $J = J_{\rm sup} = J[\phi]$,

$$\mathcal{G}^a(p) \left[\Gamma_k - \int \left(\frac{1}{2\alpha} (\partial_\mu A_\mu^a)^2 + (\partial_\mu \bar{c}^a) D_\mu^{ab}(A) c^b \right) \right]$$

$$
\begin{aligned}
&= -\frac{1}{\alpha} g f^{abc} \int_q (p+q)_\mu (p+q)_\nu \langle A^c_\mu(-q) A^b_\nu(p+q)\rangle_{\mathrm{con}} \\
&\quad - g f^{abc} \int_{q_1,q_2} p_\mu \big(\delta(p+q_1-q_2) q_{2\mu} \delta^{bf} - \mathrm{i} g f^{bef} A^e_\mu(p+q_1-q_2)\big) \\
&\qquad\qquad \times \langle \bar{c}^c(q_1) c^f(q_2)\rangle_{\mathrm{con}} \\
&\quad - \mathrm{i} g^2 f^{abc} f^{feb} \int_{q_1,q_2} p_\mu \langle \bar{c}^c(q_1) A^e_\mu(p+q_1-q_2) c^f(q_2)\rangle_{\mathrm{con}}, \qquad (89)
\end{aligned}
$$

where $\langle\ldots\rangle_{\mathrm{con}}$ denotes only the connected part of the correlation functions, e.g., $\langle\varphi\varphi\rangle_{\mathrm{con}} = \langle\varphi\varphi\rangle - \langle\varphi\rangle\langle\varphi\rangle$. All terms on the right-hand side are loop terms, the last term is even a two-loop term. It should also be stressed that the WTI is expressed here in terms of unrenormalized fields and couplings.

As an example, let us see how the WTI imposes constraints on operators in the effective action. For this, we discuss a gluon mass term in the Landau gauge, $\alpha \to 0$. The gluon-mass operator reads,

$$
\Gamma_{\mathrm{mass}} = \frac{1}{2} \int m_A^2\, A^a_\mu A^a_\mu. \qquad (90)
$$

Its gauge transform yields

$$
\mathcal{G}^a(p) \Gamma_{\mathrm{mass}} = m_A^2\, \mathrm{i} p_\mu A^a_\mu(p). \qquad (91)
$$

This implies that we have to project the remaining terms of the WTI only onto the operator $\sim p_\mu A^a_\mu(p)$ in order to study the gauge constraint on the gluon mass. Let us do so for the first term on the right-hand side of Eq. (89) by way of example; shifting the momentum q by $q \to q - p$, this term reads

$$
\begin{aligned}
&-\frac{1}{\alpha} g f^{abc} \int_q q_\mu q_\nu \langle A^c_\mu(p-q) A^b_\nu(q)\rangle_{\mathrm{con}}|_{\alpha\to 0} \\
&\qquad = -\frac{1}{\alpha} \int_q P_{\mathrm{L},\mu\nu}(q)\, q^2\, G^{cb}_{\mu\nu}(p-q, q|A, c, \bar{c})|_{\alpha\to 0}, \qquad (92)
\end{aligned}
$$

where we have introduced the longitudinal projector $P_{\mathrm{L},\mu\nu} = q_\mu q_\nu / q^2$ as well as the full gluon propagator in the background of all fields $G^{cb}_{\mu\nu}(p-q, q|A, c, \bar{c})$. In view of Eq. (91), this propagator is needed only to linear order in the gauge field,

$$
\begin{aligned}
G_{\mu\nu}(p-q, q|A, c, \bar{c}) = {}& G_{\mu\nu}(p-q, q) \\
& + G_{\mu\kappa}(p-q, q-p) V_{3A,\kappa\lambda\rho}(p-q, -p, q) G_{\rho\nu}(-q, q) A_\lambda(p) \\
& + \mathcal{O}(A^2, \bar{c}c), \qquad (93)
\end{aligned}
$$

where V_{3A} denotes the full three-gluon vertex, and all momenta are counted as in-flowing. Inserting the order linear in A of Eq. (93) into Eq. (92), we observe that both gluon propagators are contracted with the longitudinal projector,

$$-\frac{1}{\alpha} g f^{abc} \int_q q_\mu q_\nu \langle A_\mu^c(p-q) A_\nu^b(q)\rangle_{\rm con}|_{\alpha\to 0}$$
$$= -\frac{1}{\alpha}\int_q q^2 G_{\mathrm{L},\mu\kappa} V_{3A,\kappa\lambda\rho} G_{\mathrm{L},\rho\mu} A_\lambda|_{\alpha\to 0}. \tag{94}$$

Now, the Landau gauge strictly enforces the gauge fields to be transverse. Any longitudinal modes have to decouple in the Landau-gauge limit; in particular, we have $G_{\mathrm{L}} \sim \alpha \to 0$. As a consequence, the whole expression (94) goes to zero linearly with α, at least order by order in perturbation theory. We conclude that this first term on the right-hand side of the WTI (89) does not support a nonvanishing value for the gluon mass. Let us mention without proof that the same property can be shown also for all other terms on the right-hand side of Eq. (89). Therefore, the WTI enforces the gluon mass term to vanish to any order in perturbation theory, $\Gamma_{\rm mass} = 0$, implying $m_A^2 = 0$. The WTI protects the zero mass of the gluon against perturbative quantum contributions, because of gauge invariance.

Let us now turn to the modifications of the gauge constraint in the presence of a regulator. We already derived the regulator-modified WTI (mWTI) in Eq. (74),

$$\mathcal{W}_k := \mathcal{G}\Gamma_k + \mathcal{G}\Delta S_k - \langle \mathcal{G}(S_{\rm gf} + S_{\rm gh} + \Delta S_k)\rangle = 0. \tag{95}$$

For an explicit representation, we need the gauge transforms of the regulator terms,

$$\mathcal{G}^a(p)\Delta S_{k,A} = i p_\mu (R_{k,A})^{ab}_{\mu\nu}(p) A_\nu^b(p) - g f^{abc}\int_q A_\mu^c(q)(R_{k,A})^{bd}_{\mu\nu}(p-q) A_\nu^d(p-q), \tag{96}$$

$$\mathcal{G}^a(p)\Delta S_{k,\rm gh} = -g f^{abc}\int_q \bar{c}^c(q)[R_{k,\rm gh}(q+p) - R_{k,\rm gh}(q)] c^b(q+p). \tag{97}$$

Using our previous result for the standard WTI (89), the mWTI can now be displayed in the more explicit form,

$$\begin{aligned}
&\mathcal{G}^a(p)\left[\Gamma_k - \int\left(\frac{1}{2\alpha}(\partial_\mu A_\mu^a)^2 + (\partial_\mu \bar{c}^a) D_\mu^{ab}(A) c^b\right)\right] \\
&\quad = -\frac{1}{\alpha} g f^{abc}\int_q (p+q)_\mu (p+q)_\nu \langle A_\mu^c(-q) A_\nu^b(p+q)\rangle_{\rm con} \\
&\quad\quad - g f^{abc}\int_{q_1,q_2} p_\mu\big(\delta(p+q_1-q_2) q_{2\mu}\delta^{bf} - \mathrm{i} g f^{bef} A_\mu^e(p+q_1-q_2)\big) \\
&\quad\quad\quad \times\langle \bar{c}^c(q_1) c^f(q_2)\rangle_{\rm con} \\
&\quad\quad - \mathrm{i} g^2 f^{abc} f^{feb}\int_{q_1,q_2} p_\mu \langle \bar{c}^c(q_1) A_\mu^e(p+q_1-q_2) c^f(q_2)\rangle_{\rm con} \\
&\quad\quad - \frac{1}{2} g f^{abc}\int_q \big[(R_{k,A})_{\mu\nu}(p+q) - (R_{k,A})_{\mu\nu}(q)\big]\langle A_\mu^c(-q) A_\nu^b(p+q)\rangle_{\rm con}
\end{aligned}$$

$$-gf^{abc}\int_q \big[R_{k,\mathrm{gh}}(p+q)-R_{k,\mathrm{gh}}(q)\big]\langle\bar{c}^c(q)c^b(q+p)\rangle_{\mathrm{con}}\quad\Big|_{\alpha\to 0}. \qquad (98)$$

The last two terms denote the modification of the mWTI in comparison to the standard WTI. These two terms are one-loop terms with a structure similar to the flow equation itself. Both terms vanish in the limit $k\to 0$ and the standard WTI is recovered, as it should. Again, we stress that the mWTI is expressed in terms of unrenormalized fields and couplings.

As an example, it would be straightforward to work out the precise contribution of the regulator terms to the gluon mass which does not vanish in contrast to the WTI result [50, 53]. However, here it suffices to estimate the order of magnitude of this contribution. The structure $R_k(p+q)-R_k(q)$, together with the fact that we need to project only on the terms linear in A and p_μ (cf. Eq. (91)), implies that the q integral is peaked around $q^2\simeq k^2$. The dimensionful scales on the right-hand side for the gluon mass are thus set by k^2, resulting in $m_A^2\sim g^2k^2$ with a proportionality coefficient that depends on the regulator. This is a very unusual bosonic-mass running which guarantees that the gluon mass is not a relevant operator in the flow, but vanishes with $k\to 0$.

Let us close this subsection by briefly discussing the connection of the present formalism with the BRST formalism. The latter involves one further conceptual step, emphasizing BRST invariance as a residual invariance of the gauge-fixed functional integral. The corresponding symmetry constraints on the effective action, the *Slavnov-Taylor identities*, have the advantage in the standard formulation that they are bilinear in derivatives of the effective action. This allows for an algebraic resolution of the gauge constraints in contrast to the loop computations necessary for the WTI, cf. Eq. (89). If the regulator term is included, modified Slavnov-Taylor identities can still be derived [19, 21, 22, 45], but the result no longer has a bilinear structure. We conclude that the BRST formulation has no real advantage in the case of the RG flow equation, such that the present formalism can fully be recommended also for practical applications.

3.5 Further reading: Landau-gauge IR propagators*

The following paragraphs give a short introduction to recent lines of research, and may serve as a guide to the literature.

The functional RG techniques for gauge theories developed above can now be used for computing the effective action in a vertex expansion, cf. Eq. (29). In momentum space, the expansion reads,

$$\Gamma_k[\phi]=\sum_n\frac{1}{n}\int_{p_1,\dots,p_n}\delta(p_1+\ldots+p_n)\,\Gamma_k^{(n)}(p_1,\dots,p_n)\,\phi(p_1)\ldots\phi(p_n), \quad (99)$$

where $\int_p=\int d^Dp/(2\pi)^D$, $\delta(p)=(2\pi)^D\delta^{(D)}(p)$, and $\phi=(A_\mu^a,\bar{c}^a,c^a)$. Inserting Eq. (99) into the flow equation (73), we obtain an infinite set of coupled first-oder differential equations for the proper vertices $\Gamma_k^{(n)}$. Truncating the expansion at order $n_{\max}$ leaves all equations for the vertices $\Gamma_k^{(n\leq n_{\max}-2)}$ unaffected. In order to close this tower of equations, the vertices of order $n_{\max}$ and $n_{\max}-1$ can either be derived from their truncated equations or taken as bare or constructed by further considerations; see, e.g., [54]. This defines a consistent approximation scheme that can in principle be iterated to arbitrarily high orders in $n_{\max}$.

Let us consider here the lowest nontrivial order,

$$\Gamma_k = \frac{1}{2}\int_q A_\mu^a(-q)\left[Z_A(q^2)q^2\, P_{\mathrm{T}\mu\nu} + m_k^2\delta_{\mu\nu}\right]A_\nu^a(q) + \int_q \bar{c}^a(q)\, Z_{\mathrm{gh}}(q^2)q^2\, c^a(q) + \dots, \tag{100}$$

where P_T is the transverse projector, and the ellipsis denotes higher-order vertices and longitudinal gluonic terms. In the following, we will confine ourselves to the Landau gauge $\alpha = 0$ where longitudinal modes decouple completely; moreover, the Landau gauge is known to be a fixed-point of the RG flow [49, 46]. The nontrivial ingredients consist in the fully momentum-dependent wave function renormalizations $Z_A(p^2)$ and $Z_{\mathrm{gh}}(p^2)$, as well as a gluon mass term which has been discussed in detail above. The flow equations for the inverse of the transverse gluon and ghost propagators ($G_k = (\Gamma_k^{(2)} + R_k)^{-1}$),

$$\Gamma_{k,A\,\mathrm{T}}^{(2)}(p^2) = Z_A(p^2)p^2 + m_k^2, \quad \Gamma_{k,\mathrm{gh}}^{(2)}(p^2) = Z_{\mathrm{gh}}(p^2)p^2, \tag{101}$$

are shown in Fig. 9, except for diagrams involving quartic vertices. The flow-equation diagrams are reminiscent to those of Dyson-Schwinger equations. But there are two differences: Dyson-Schwinger equations also involve two-loop diagrams, whereas the flow equation imposes its one-loop structure also on the propagator and vertex equations. Second, all internal propagators and vertices are fully dressed quantities in the flow equation, whereas Dyson-Schwinger equations always involve one bare vertex. Both vertex expansions are slightly different infinite-tower expansions of the same generating functional, with the flow equations being amended by the differential RG structure.

A truncation at this order requires information about the triple and quartic vertices. In a minimalistic approach, they may be taken as bare (possibly accompanied by k-dependent renormalization constants). In general, this non-perturbative approximation is expected to be reliable at weak coupling. For instance, the perturbative result is rediscovered at high scales, $k = \Lambda_\mathrm{p}$; for momenta p^2 larger than this perturbative scale Λ_p^2, the wave function renormalizations yield,

$$\begin{aligned} Z_A(p^2) &\simeq Z_{\Lambda_\mathrm{p},A}\left(1 - \eta_A\,\frac{11 N_\mathrm{c}\alpha_{\Lambda_\mathrm{p}}}{12\pi}\ln\frac{p^2}{\Lambda_\mathrm{p}^2}\right), \\ Z_{\mathrm{gh}}(p^2) &\simeq Z_{\Lambda_\mathrm{p},\mathrm{gh}}\left(1 - \eta_{\mathrm{gh}}\,\frac{11 N_\mathrm{c}\alpha_{\Lambda_\mathrm{p}}}{12\pi}\ln\frac{p^2}{\Lambda_\mathrm{p}^2}\right), \end{aligned} \tag{102}$$

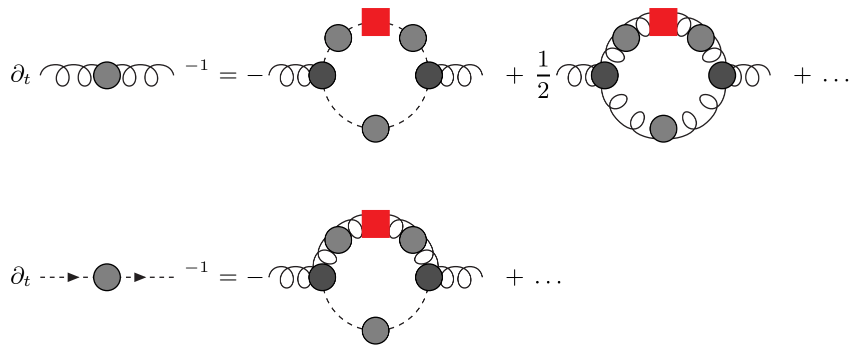


Fig. 9. Flow equations for gluon and ghost propagators in a vertex expansion. All internal lines and vertices denote fully dressed quantities, indicated by filled circles. To each diagram, there is another one with identical topology but with the regulator insertion occurring at the opposite internal line. The ellipses denote diagrams involving quartic vertices which are not displayed.

where $\eta_A = -13/22$ and $\eta_{\rm gh} = -9/44$ denote the anomalous dimensions for gluons and ghosts, respectively. $Z_{\Lambda_{\rm p},A}, Z_{\Lambda_{\rm p},{\rm gh}}$ are the normalizations of the fields, and $\alpha_{\Lambda_{\rm p}}$ is the value of the coupling constant at $\Lambda_{\rm p}$.

At first glance, there is no reason why the higher-order vertex structures which are dropped in the minimalistic truncation should not become dominant at strong coupling. However, as we learned from the anharmonic oscillator example in Subsect. 2.3, the quality of a low-order truncation does not depend on how large higher-order terms may get but whether they exert a strong influence on the low-order equations. Moreover, mechanisms may exist that systematically suppress higher-order contributions, such as an IR suppression of certain propagators; since higher-order vertex equations involve more propagators, such a propagator suppression would control large classes of diagrams.

In recent years, evidence has been provided that such a suppression is indeed operative in the gluon sector in the Landau gauge: low-order vertex expansions reveal an IR suppressed gluon propagator which renders the contributions from higher gluonic vertices subdominant. This solution has been pioneered in [55] using truncated Dyson-Schwinger equations, see [25, 28] for reviews; IR gluon suppression in the Landau gauge has meanwhile been confirmed by many lattice simulations [56, 57, 58, 59, 60, 61, 62, 63]. In the continuum, this scenario goes along with an IR enhanced ghost propagator, i.e., IR *ghost dominance*. This IR enhancement does not spoil the vertex expansion owing to a nonrenormalization theorem for the ghost-gluon vertex [64], the running of which is thus protected against strong renormalization effects. Also ghost dominance has been observed on the lattice [65, 61], even though Gribov-copy and/or finite-volume/size effects appear to affect the IR ghost sector more strongly.

The nonrenormalization theorem of the ghost-gluon vertex in the Landau gauge gives rise to a nonperturbative definition of the running coupling in terms of the wave function renormalizations,

$$\alpha(p^2) = \frac{g^2}{4\pi} \frac{1}{Z_A(p^2)\, Z_{\mathrm{gh}}^2(p^2)}. \tag{103}$$

In the IR, the above-described scenario of ghost dominance and gluon suppression is quantitatively observed in terms of a power-law behavior of the wave function renormalizations,[8]

$$Z_A(p^2) \sim (p^2)^{-2\kappa}, \quad Z_{\mathrm{gh}}(p^2) \sim (p^2)^{\kappa}, \tag{104}$$

where κ denotes a positive IR exponent. In the functional RG framework, this solution can be shown to be a fixed point of the flow equations for the propagators, cf. Fig. 9, in the momentum regime $k^2 \ll p^2 \ll \Lambda_{\mathrm{QCD}}^2$ [66]. The interrelation of the ghost and gluon propagators owing to the simultaneous occurrence of the exponent κ arises in all functional approaches from self-consistency arguments; as a direct consequence, the running coupling (103) approaches a fixed point in the IR, $\alpha(p^2 \ll \Lambda_{\mathrm{QCD}}^2) \to \alpha_*$. For instance, a truncation with a bare ghost-gluon vertex results in $\kappa \simeq 0.595$; possibly induced momentum dependencies of the vertex can lead to slightly lower values $0.5 \leq \kappa \leq 0.595$ [67]. Regulator dependencies arising in an RG calculation also lie in this range [68]. The IR solutions (104) are IR attractive fixed-point solutions for a wide class of initial conditions and momentum-dependencies of the gluon vertices, once the gluon propagator has developed a mass-like structure at intermediate momenta at a few times Λ_{QCD} [53]. Using suitable vertex *ansätze*, full solutions connecting the perturbative UV branch Eq. (102) and the IR power-laws (104) have been found with Dyson-Schwinger equations, see [69, 28].

Most importantly, the IR power-law behavior featuring gluon suppression and ghost dominance agrees with criteria which are expected to be satisfied in two different scenarios of confinement: the Kugo-Ojima [70] and the Gribov-Zwanziger [41, 71, 72] confinement scenario. In particular, a strongly IR divergent ghost propagator represents a signature of confinement in these scenarios, which describe the absence of color charged asymptotic states and (indirectly) a linear rise of the potential between a static quark-antiquark pair. The study of correlation functions in connection with these scenarios of low-energy gauge theories clearly demonstrates the potential of functional methods to access even the strongly-coupled gauge sector by analytical means.

[8] In the Dyson-Schwinger literature, the gluon and ghost propagator behavior is often characterized by *dressing functions* $Z_{\mathrm{DSE}}, G_{\mathrm{DSE}}$ which are related to the wave function renormalizations by $Z_A(p^2) = Z_{\mathrm{DSE}}^{-1}(p^2)$ and $Z_{\mathrm{gh}}(p^2) = G_{\mathrm{DSE}}^{-1}(p^2)$ for $k \to 0$.

4 Background-field flows

Imagine for a second that we knew nothing about computing loops and constructing amplitudes in some sort of expansion which involves a perturbative or even a fully dressed propagator. If we knew only the degrees of freedom of our gauge system and the symmetries we would be trying to write down an effective action in terms of all possible gauge-invariant gluon operators, such as $F^a_{\mu\nu}F^a_{\mu\nu}$, $F^a_{\mu\nu}(D_\kappa D_\kappa)^{ab}F^b_{\mu\nu}$ etc. and determine the coefficients, e.g., in a manner similar to chiral perturbation theory. The result would be gauge invariant by construction, and we would never worry about Ward identities and how gauge invariance can be encoded in a nontrivial manner in a gauge-fixed formulation.

4.1 Background-field formalism

The background-field formalism aims precisely at the construction of such an effective action, nevertheless by computing loops and integrating out fluctuations in a special gauge-fixed manner. Here is a rough sketch of the idea:

(1) Introduce an auxiliary, non-dynamical field $\bar{A}^a_\mu$ (background field) with its own auxiliary symmetry transformation $\bar{\mathcal{G}}$.
(2) Construct a gauge-fixed QFT which has broken invariance under the standard gauge transformation $\mathcal{G}$ but a manifest invariance under $(\mathcal{G}+\bar{\mathcal{G}})$.
(3) Let the full Γ inherit the symmetry properties in the end by setting $A=\bar{A}$ after the gauge-fixed calculation.

Let us start with (1): we introduce $\bar{A}^a_\mu$ and corresponding covariant derivatives $\bar{D}^{ab}_\mu = \partial_\mu\delta^{ab} + gf^{acb}\bar{A}^c_\mu$, and the generator of symmetry transformations

$$\bar{\mathcal{G}}^a(x) = \bar{D}^{ab}_\mu \frac{\delta}{\delta \bar{A}^b_\mu}, \tag{105}$$

which we call the background transformation. Note that the ghosts are not affected by $\bar{\mathcal{G}}$. Together with Eq. (69), it is obvious that $(A-\bar{A})$ now transforms homogeneously under $\mathcal{G}+\bar{\mathcal{G}}$,

$$\int \mathrm{d}^D y\, \omega^b(y)(\mathcal{G}+\bar{\mathcal{G}})^b(y)(A-\bar{A})^a_\mu(x) = gf^{abc}\omega^b(x)(A-\bar{A})^c_\mu(x). \tag{106}$$

As step (2), we choose a gauge fixing F^a which fixes the $\mathcal{G}$ symmetry but is invariant under $\mathcal{G}+\bar{\mathcal{G}}$:

$$\begin{aligned}\mathsf{F}^a &= \bar{D}^{ab}_\mu (A^b_\mu - \bar{A}^b_\mu)\\ &\Rightarrow (\mathcal{G}+\bar{\mathcal{G}})S_{\mathrm{gf}} = \frac{1}{2\alpha}\,(\mathcal{G}+\bar{\mathcal{G}})\int \mathrm{d}^D x\,[\bar{D}(A-\bar{A})]^2 = 0.\end{aligned} \tag{107}$$

In fact, the gauge-fixing term is invariant under the combined transformation. With the Faddeev-Popov operator

$$\frac{\delta \mathsf{F}^a}{\delta \omega^b} = -\bar{D}_\mu^{ac} D_\mu^{cb}, \tag{108}$$

it is also straightforward to show that

$$(\mathcal{G} + \bar{\mathcal{G}}) S_{\mathrm{gh}} = -(\mathcal{G} + \bar{\mathcal{G}}) \int \mathrm{d}^D x \, \bar{c}^a \bar{D}_\mu^{ac} D_\mu^{cb} c^b = 0. \tag{109}$$

Finally, let us merely sketch step (3): the price to be paid so far is that Γ now depends on A and $\bar{A}$. But at the end of the calculation, we can identify $A = \bar{A}$, such that

$$0 = (\mathcal{G} + \bar{\mathcal{G}}) \Gamma[A, \bar{A}] \big|_{A=\bar{A}} = \mathcal{G} \Gamma[A, A], \tag{110}$$

where the first equality holds by construction and the second arises from setting $A = \bar{A}$. Now, it is possible to prove that the background effective action with $A = \bar{A}$ reduces precisely to the standard effective action, $\Gamma[A, A] \equiv \Gamma[A]$ [73, 24, 74]. Therefore, Eq. (110) verifies the desired gauge-invariant construction of $\Gamma[A]$. This $\Gamma[A]$ thus only consists of gauge-invariant building blocks. Of course, there is still a nontrivial constraint which becomes visible if we go away from the limit $A = \bar{A}$; namely, $\mathcal{G}\Gamma[A, \bar{A}]$ has to satisfy the standard WTI [47].

4.2 Background-field flow equation

The desired properties of the effective action expressed by Eq. (110) can be maintained in the construction of the corresponding RG flow equation, if also the regulator satisfies

$$(\mathcal{G} + \bar{\mathcal{G}}) \Delta S_k = 0. \tag{111}$$

This holds, e.g., for the choice

$$\Delta S_k = \frac{1}{2} \int (A - \bar{A}) R_{k,A}(\bar{\Delta}_A)(A - \bar{A}) + \int \bar{c} R_{k,\mathrm{gh}}(\bar{\Delta}_{\mathrm{gh}}) c, \tag{112}$$

where $\bar{\Delta}_{A,\mathrm{gh}}$ are operators that can depend on $\bar{A}$ and transform homogeneously. For the gluon sector, a suitable choice can, for instance, be given by

$$(\bar{\Delta}_A)^{ac}_{\mu\nu} = \{-\bar{D}_\kappa^{ab} \bar{D}_\kappa^{bc} \delta_{\mu\nu}, \, -\bar{D}_\kappa^{ab} \bar{D}_\kappa^{bc} \delta_{\mu\nu} + 2\mathrm{i}g(\bar{F}^b_{\mu\nu} T^b)^{ac}, \ldots\}, \tag{113}$$

where the first form corresponds to the background-covariant Laplacian, and the second also contains the spin-one coupling to the background field. For the ghost sector, the Laplacian is also an option, $\bar{\Delta}_{\mathrm{gh}} = -\bar{D}_\kappa^{ab} \bar{D}_\kappa^{bc}$. The resulting flow equation in the background-field gauge reads [75]

$$\partial_t \Gamma_k[A, \bar{c}, c, \bar{A}] = \frac{1}{2} \, \mathrm{STr} \left\{ \partial_t R_k(\bar{\Delta}) [\Gamma_k^{(2)}[A, \bar{A}] + R_k(\bar{\Delta})]^{-1} \right\}. \tag{114}$$

Here, it is a temptation to set $A = \bar{A}$ in the flow equation; however, the above construction tells us that this should be done only at the end of the calculation at $k = 0$.

Let us parameterize (suppressing ghosts for a moment) the effective action as [75]

$$\Gamma_k[A, \bar{A}] = \Gamma_k^{\text{inv}}[A] + \Gamma_k^{\text{gauge}}[A, \bar{A}], \tag{115}$$

where Γ_k^{inv} is a gauge-invariant functional, and Γ_k^{gauge} denotes the gauge-non-invariant remainder that satisfies $\Gamma_k^{\text{gauge}}[A, \bar{A} = A] = 0$, cf. Eq. (110). Considering the second functional derivative with respect to A,

$$\Gamma_k^{(2)}[A, \bar{A}] = \Gamma_k^{\text{inv}(2)}[A] + \Gamma_k^{\text{gauge}(2)}[A, \bar{A}], \tag{116}$$

we observe that $\Gamma_k^{\text{inv}(2)}[A]$ must be singular. This is because of the zero modes associated with gauge invariance: a variation with respect to A which points tangentially to the gauge orbit has to leave Γ_k^{inv} invariant, corresponding to a flat direction. For the flow equation (114) to be well defined, the contribution of $\Gamma_k^{\text{gauge}(2)}$ to the denominator in Eq. (114) has to lift these flat directions. This demonstrates that Γ_k^{gauge} must not be dropped from a truncation, even though we may ultimately be interested only in Γ_k^{inv}.

On the other hand, the mWTI does not impose any constraint on Γ_k^{inv},

$$0 = \mathcal{W}_k[\Gamma_k] \equiv \mathcal{W}_k[\Gamma_k^{\text{gauge}}], \tag{117}$$

which implies that we have the full freedom to choose any gauge-invariant functional as an ansatz for Γ_k^{inv}, and its solution will solely be determined by the flow equation.

To summarize, the background formalism facilitates the construction of a gauge-invariant RG flow in which the manifestly gauge-invariant parts of the effective action Γ_k^{inv} can be separated from the gauge-dependent parts Γ_k^{gauge}. In practice, we can construct our ansatz for Γ_k^{inv} by picking gauge-invariant building blocks, in a similar manner as we would do for other effective field theories. In addition, we have to construct (a truncation for) Γ_k^{gauge} with the aid of the mWTI (117), in order to lift the gauge zero modes in the flow equation.

4.3 Running coupling

The background formalism also provides for a convenient nonperturbative definition of the running coupling. As a general remark, let us stress that there is no unique definition of the running coupling in the nonperturbative domain. Universality of the running coupling holds only near fixed points; e.g., only the one-loop coefficient of the perturbative β function is definition and scheme independent (in a mass-independent regularization scheme, also the two-loop coefficient is universal). Hence, any result for the running coupling in the nonperturbative domain has to be understood strictly in the context of its definition.

The definition within the background formalism follows, for instance, from the RG invariance of the background-covariant derivative, which is a gauge-covariant building block of Γ_k^{inv},

$$\bar{D}_\mu^{ab}[\bar{A}] = \partial_\mu \delta^{ab} + \bar{g} f^{abc} \bar{A}_\mu^c. \tag{118}$$

Here, we have used the notation $\bar{g}$ for the bare coupling. Obviously, the first term $\partial_\mu \delta^{ab}$ is RG invariant. Hence, also the product of $\bar{g}$ and $\bar{A}$ must be RG invariant [73],

$$\partial_t (\bar{g} \bar{A}_\mu^c) = 0, \quad \Rightarrow \quad \bar{g} \bar{A}_\mu^c = g \bar{A}_{\mathrm{R},\mu}^c, \tag{119}$$

where g now denotes the renormalized coupling and $\bar{A}_{\mathrm{R}}$ the renormalized background field. Renormalization of the background field is described by a wave function renormalization factor, $\bar{A}_{\mathrm{R}} = Z_k^{1/2} \bar{A}$. Consequently, also the running of the coupling is tied to the same wave function renormalization,

$$g^2 = Z_k^{-1} \bar{g}^2. \tag{120}$$

We obtain for the β function of the running coupling

$$\beta_{g^2} \equiv \partial_t g^2 = \eta g^2, \quad \eta = -\partial_t \ln Z_k, \tag{121}$$

where η denotes the anomalous dimension of the background field. The wave function renormalization of the background field can be read off from the kinetic term of the background gauge potential,

$$\Gamma_k^{\text{inv}}[A] = \int \frac{Z_k}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \dots, \tag{122}$$

where the dots represent further terms in the truncation. The running coupling is thus linked to the lowest-order term of an operator expansion of the effective action.

According to its definition, the running coupling can be viewed as the response coefficient to excitations about the background field. If the background field, as a natural choice, is associated with the vacuum state the running coupling measures the coupling between the vacuum and excitations, for instance, in the form of (effective) quark and gluon fluctuations.

4.4 Truncated background flows

Let us study the background flow in a simple approximation, following [76]. In particular, we will be satisfied by a minimal truncation for Γ_k^{gauge}. For this, we expand Γ^{gauge} to leading order in $A - \bar{A}$,

$$\Gamma_k^{\text{gauge}}[A, \bar{A}] = \int (A - \bar{A})_\mu^a \, M_{\mu\nu}^{ab} \, (A - \bar{A})_\nu^b + \mathcal{O}((A - \bar{A})^3). \tag{123}$$

Then, we fix $M^{ab}_{\mu\nu}$ by the tree-level order of the mWTI:

$$\Gamma_k^{\mathrm{gauge}}[A,\bar{A}] = \frac{1}{2\alpha}\int (A-\bar{A})^a_\mu \, (-\bar{D}^{ac}_\mu \bar{D}^{cb}_\nu)\,(A-\bar{A})^b_\nu + \dots, \tag{124}$$

which is just the classical gauge-fixing term. This approximation has an important consequence: to this order,

$$\Gamma_k^{(2)} = \Gamma_k^{\mathrm{inv}(2)} + \Gamma_k^{\mathrm{gauge}(2)} \tag{125}$$

is independent of $(A-\bar{A})$, and we can set $A=\bar{A}$ under the flow for all k. The form of $\Gamma_k^{\mathrm{gauge}}$ in Eq. (124) is just enough to lift the gauge zero modes. This gives us an approximate flow for Γ_k^{inv},

$$\begin{aligned}\partial_t \Gamma_k^{\mathrm{inv}}[A] = \frac{1}{2}\,\mathrm{Tr}\left\{\partial_t R_{k,A}(\Delta_A)\left[\Gamma_k^{\mathrm{inv}(2)} + \frac{1}{\alpha}(-DD) + R_k(\Delta)\right]^{-1}_A\right\}\\ -\mathrm{Tr}\left\{\partial_t R_{k,\mathrm{gh}}(\Delta_{\mathrm{gh}})\left[\Gamma_k^{\mathrm{inv}(2)} + R_k(\Delta)\right]^{-1}_{\mathrm{gh}}\right\},\end{aligned} \tag{126}$$

where we have dropped the bars on the right-hand side, since $A=\bar{A}$. So far, we have suppressed the ghost fields $c,\bar{c}$. If the dependence of Γ_k^{inv} on the ghosts is such that ghosts are contracted with homogeneously transforming color tensors, the mWTI is satisfied to the same level of accuracy as by the ghost-independent part. The classical ghost action in background-field gauge reads $S_{\mathrm{gh}} = -\int \mathrm{d}^D x\, \bar{c}^a \bar{D}^{ab} D^{bc} c^c$; hence, the lowest-order approximation for our truncation is given by the classical term at $A=\bar{A}$,

$$\Gamma_{k,\mathrm{gh}}^{\mathrm{inv}}[A,\bar{c},c] = -\int \mathrm{d}^D x\, \bar{c}^a D^{ab} D^{bc} c^c. \tag{127}$$

Let us now concentrate on the running coupling in the background gauge. As discussed above, this can be read off from the kinetic terms of the gauge field which we will therefore choose as the only nontrivial part of our simplest nontrivial truncation,

$$\Gamma_{k,\mathrm{glue}}^{\mathrm{inv}}[A] = \int \frac{Z_k}{4}\, F^a_{\mu\nu} F^a_{\mu\nu}, \tag{128}$$

where the running of the wave function renormalization remains to be calculated. This is done by projecting the right-hand side of the flow equation onto this kinetic operator only; any other operator generated by the flow, e.g., containing ghost fields, is dropped. Hence, we can set $c,\bar{c}=0$ in $\Gamma^{(2)}$ (of course, *after* functional differentiation), implying that $\Gamma^{(2)}$ becomes block-diagonal with respect to gluon and ghost sectors. In the gluon sector, we obtain

$$\delta^2 \Gamma_k^{\mathrm{inv}}|_{\mathrm{glue}} = Z_k \int \mathrm{d}^D x\, \delta A^a_\mu \left[\mathcal{D}^{ab}_{\mathrm{T},\mu\nu} + D^{ac}_\mu D^{cb}_\nu - \frac{1}{\alpha} D^{ac}_\mu D^{cb}_\nu\right] \delta A^b_\nu, \tag{129}$$

where we have introduced the notation

$$\mathcal{D}^{ab}_{\mathrm{T},\mu\nu} = -D^{ac}_\kappa D^{cb}_\kappa \delta_{\mu\nu} + 2\bar{g} f^{abc} F^c_{\mu\nu}, \tag{130}$$

for the covariant spin-one Laplacian. Using the Feynman gauge $\alpha = 1$ here for simplicity, the gluon sector thus reduces to

$$(\Gamma_k^{\mathrm{inv}(2)})^{ab}_{\mu\nu}|_{\mathrm{glue}} = Z_k \mathcal{D}^{ab}_{\mathrm{T},\mu\nu}. \tag{131}$$

From Eq. (127), we can immediately read off the form of $\Gamma^{(2)}$ in the ghost sector,[9]

$$\delta^2 \Gamma_k^{\mathrm{inv}}|_{\mathrm{gh}} = -\int \mathrm{d}^D x\, \delta\bar{c}^a D^{ac} D^{cb} \delta c^b, \quad \Rightarrow \quad (\Gamma_k^{\mathrm{inv}(2)})^{ab}|_{\mathrm{gh}} = -D^{ac}_\kappa D^{cb}_\kappa. \tag{132}$$

Upon insertion into Eq. (126), the flow equation boils down to

$$\partial_t \Gamma_k^{\mathrm{inv}}[A] = \frac{1}{2}\,\mathrm{Tr}\,\left[\partial_t R_{k,A}(Z_k \mathcal{D}_\mathrm{T} + R_{k,A})^{-1}\right] - \mathrm{Tr}\,\left[\partial_t R_{k,\mathrm{gh}}(-D^2 + R_{k,\mathrm{gh}})^{-1}\right]. \tag{133}$$

A convenient choice for the regulator is given by (cf. Eq. (16))

$$R_{k,A} = Z_k\, \mathcal{D}_\mathrm{T}\, r(\mathcal{D}_\mathrm{T}/k^2), \quad R_{k,\mathrm{gh}} = (-D^2)\, r(-D^2/k^2), \quad \lim_{y\to 0} r(y) \to \frac{1}{y}. \tag{134}$$

The insertion of the wave function renormalization Z_k into the regulator is useful, since it maintains the invariance of the flow equation under RG rescalings of the fields (the ghost wave function renormalization has been set to $Z_{k,\mathrm{gh}} = 1$ in our truncation). The choice $r(y) \to \frac{1}{y}$ implies that the IR modes with $p^2 \lesssim k^2$ are regulated by acquiring a mass term $\sim k^2$; the identification of the one-loop running is more straightforward with this choice. Then, the flow equation reads

$$\begin{aligned} \partial_t \Gamma_k^{\mathrm{inv}}[A] \;&=\; \frac{1}{2}\,\mathrm{Tr}\,\left[\frac{\partial_t (Z_k r(\mathcal{D}_\mathrm{T}/k^2))}{Z_k(1 + r(\mathcal{D}_\mathrm{T}/k^2))}\right] - \mathrm{Tr}\,\left[\frac{\partial_t r(-D^2/k^2)}{1 + r(-D^2/k^2)}\right] \\ &=: \mathrm{Tr}\,\mathcal{H}_{Z_k}(\mathcal{D}_\mathrm{T}/k^2) - 2\,\mathrm{Tr}\,\mathcal{H}(-D^2/k^2). \end{aligned} \tag{135}$$

The obvious definition of the $\mathcal{H}$ functions in the last line expresses the fact that both terms on the right-hand side are traces over a function of a single operator. It is useful to formally introduce the Laplace transform of the $\mathcal{H}$ functions,

$$\mathcal{H}(y) = \int_0^\infty ds\, \widetilde{\mathcal{H}}(s)\, \mathrm{e}^{-ys}, \tag{136}$$

such that the flow equation can be written as

[9] Be aware of footnote 4 on page 7.

$$\partial_t \Gamma_k^{\mathrm{inv}}[A] = \int_0^\infty ds\, \widetilde{\mathcal{H}}_{Z_k}(s)\, \mathrm{Tr}\, \mathrm{e}^{-s(\mathcal{D}_\mathrm{T}/k^2)} - 2\int_0^\infty ds\, \widetilde{\mathcal{H}}(s)\, \mathrm{Tr}\, \mathrm{e}^{-s(-D^2/k^2)}. \tag{137}$$

As a result, we have brought the flow equation into *propertime* form. Note that this was possible because of the particular operator structure resulting from our truncation: we were able to choose the regulators such that they depend on the operators which appear as entries of the block-diagonal $\Gamma^{(2)}$. Such a propertime form of the flow can, for instance, always be established to leading order in a derivative expansion of the RG flow [77]. In fact, a wide class of truncations can be mapped onto a propertime form [29, 78] which is computationally advantageous; moreover, propertime flows have extensively and successfully been used in the literature [79, 80, 81, 82, 83, 84]. In the present case, we are finally dealing with traces over operator exponentials, so-called *heat kernels*, which can conveniently be dealt with by standard methods. The heat kernels of the covariant Laplacians occurring above have frequently been studied in the literature; see, e.g., [78]; here we merely need the term quadratic in the field strength,

$$\mathrm{Tr}\, \mathrm{e}^{-s(\mathcal{D}_\mathrm{T}/k^2)}|_{F^2} = \frac{N_\mathrm{c}(24-D)}{3} \frac{\bar{g}^2}{(4\pi)^{D/2}} \left(\frac{s}{k^2}\right)^{2-(D/2)} \int \mathrm{d}^D x \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a, \tag{138}$$

$$\mathrm{Tr}\, \mathrm{e}^{-s(-D^2/k^2)}|_{F^2} = -\frac{N_\mathrm{c}}{3} \frac{\bar{g}^2}{(4\pi)^{D/2}} \left(\frac{s}{k^2}\right)^{2-(D/2)} \int \mathrm{d}^D x \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a, \tag{139}$$

Let us from now on concentrate on four dimensional spacetime, $D = 4$. From Eq. (137), we can extract the flow of the wave function renormalization

$$\begin{aligned} \partial_t Z_k &= \frac{20 N_\mathrm{c}}{3} \frac{\bar{g}^2}{(4\pi)^2} \int_0^\infty ds\, \widetilde{\mathcal{H}}_{Z_k}(s) + \frac{2N_\mathrm{c}}{3} \frac{\bar{g}^2}{(4\pi)^2} \int_0^\infty ds\, \widetilde{\mathcal{H}}(s) \\ &= \frac{20 N_\mathrm{c}}{3} \frac{\bar{g}^2}{(4\pi)^2} \mathcal{H}_{Z_k}(0) + \frac{2N_\mathrm{c}}{3} \frac{\bar{g}^2}{(4\pi)^2} \mathcal{H}(0), \end{aligned} \tag{140}$$

where we have used Eq. (136) for $y = 0$. Together with $\partial_t (Z_k r(\mathcal{D}_\mathrm{T}/k^2)) = -Z_k [2\mathcal{D}_\mathrm{T}/k^2 r'(\mathcal{D}_\mathrm{T}/k^2) + \eta\, r(\mathcal{D}_\mathrm{T}/k^2)]$, and the anomalous dimension $\eta = -\partial_t \ln Z_k$ as defined in Eq. (121), we observe that the $\mathcal{H}$ functions for zero argument uniquely yield

$$\mathcal{H}_{Z_k}(0) = 1 - \frac{\eta}{2}, \quad \mathcal{H}(0) = 1, \tag{141}$$

independently of the precise form of the regulator shape function $r(y)$ introduced in Eq. (134). This is a direct manifestation of the regularization-scheme independence of the one-loop β function coefficient, as will become clear soon.

Introducing the renormalized coupling $g^2 = Z_k^{-1} \bar{g}^2$, cf. Eq. (120), we obtain

$$-\eta = \frac{22 N_\mathrm{c}}{3} \frac{g^2}{(4\pi)^2} - \frac{10 N_\mathrm{c}}{3} \frac{g^2}{(4\pi)^2}\, \eta. \tag{142}$$

This brings us to our final result for the Yang-Mills β function

$$\beta_{g^2} \equiv \partial_t g^2 = \eta g^2 = -\frac{22N_\mathrm{c}}{3}\frac{g^4}{(4\pi)^2}\left(1-\frac{10N_\mathrm{c}}{3}\frac{g^2}{(4\pi)^2}\right)^{-1} \tag{143}$$

$$= -\frac{22N_\mathrm{c}}{3}\frac{g^4}{(4\pi)^2} - \frac{220N_\mathrm{c}^2}{9}\frac{g^6}{(4\pi)^4} - \dots \tag{144}$$

It is instructive to compare our result to the full perturbative two-loop β function:

$$\beta_{g^2}^{2-\mathrm{loop}} = -\frac{22N_\mathrm{c}}{3}\frac{g^4}{(4\pi)^2} - \frac{204N_\mathrm{c}^2}{9}\frac{g^6}{(4\pi)^4} - \dots \tag{145}$$

Obviously, we have rediscovered the one-loop result exactly, as we should. Moreover, the two-loop coefficient comes out remarkably well with an error of only 8%. This is astonishing in two ways: first, we have dropped a number of operators that contribute to the two-loop coefficient coupling, e.g., $(F_{\mu\nu}^a F_{\mu\nu}^a)^2$ or $(F_{\mu\nu}^a \widetilde{F}_{\mu\nu}^a)^2$, and also the ghost wave function renormalization, which hence appear to be less relevant. Second, whereas Eq. (145) is a universal result obtained within a mass-independent regularization scheme such as $\overline{\mathrm{MS}}$, our result arises from a mass-dependent regularization scheme with a mass scale k; for such a scheme, the two-loop coefficient generally is not universal. However, in the present truncation, our β function indeed is universal to all orders computed above, owing to the fact that the dependence of the regulator shape function drops out in Eq. (141). We conclude that this simple truncation contains already much relevant information about the universal part of this two-loop coefficient. Larger truncations indeed contribute further terms which are partly non-universal, as expected. An exact two-loop calculation based on the functional RG can be found in [85].

In view of the quality of this simple truncation, it is tempting to speculate whether the result also contains reliable nonperturbative information. However, our result for the β function in Eq. (143) develops a pole at $g^2 = 3(4\pi)^2/(10N_\mathrm{c})$, clearly signaling the breakdown of the truncation. This could already have been expected from the physics content of the truncation: independently of the coupling strength, the resulting quantum equations of motion are identical to classical Yang-Mills theory. This equations allow for plain-wave solutions, describing freely propagating gluons. The truncation misses the important IR phenomenon of confinement and thus cannot be reliable in the deeply nonperturbative domain. For a stable flow from the UV down to the IR, a larger truncation is required that in addition has the potential to describe the relevant degrees of freedom for confinement.

4.5 Further reading: IR running coupling*

The nonperturbative estimate of the β function for the running coupling derived above has a remarkable property. All higher-loop order corrections can be summed into a geometric series, resulting in the structure of Eq. (143). The origin of this structure lies in the fact that we have included the wave function renormalization Z_k in the regulator, cf. Eq. (134). The apparent formal reason was that the RG invariance of the flow equations is thus maintained; but in addition, we thereby obtain a result which contains a resummation of a larger class of perturbative diagrams.

From a more physical viewpoint, the wave function renormalization describes the deformation of the perturbative spectrum of fluctuations, $S^{(2)} \sim p^2$, as it is induced by quantum fluctuations; at lower scales, we find the spectrum $\Gamma_k^{(2)} \sim Z_k p^2$. Therefore, the inclusion of Z_k in the regulator leads to a better adjustment of the regulator to the deformation of the spectrum, i.e., to the *spectral flow* of $\Gamma_k^{(2)}$.

If Z_k had not been included in the regulator, we would have found only the one-loop β function in the truncation (128) without any higher-loop orders. The latter would have been encoded in the flow of higher-order operators outside this simple truncation. For instance, the inclusion of the higher-order operator $(F_{\mu\nu}^a F_{\mu\nu}^a)^2$ would have resulted in a β-function estimate of the form $\beta_{g^2} = \eta g^2$, with

$$\eta = -b_0 \frac{g^2}{(4\pi)^2} - b_1 \frac{g^2}{(4\pi)^2}\, w_2, \tag{146}$$

with some coefficient b_1, and b_0 being the correct one-loop result, and w_2 denoting the generalized coupling of this higher-order operator. In this truncation, all nonperturbative information is contained in the flow of w_2, which in turn can reliably be computed only by including even higher-order operators. A good estimate therefore probably requires a very large truncation. Even if the precise infrared values of the higher couplings $w_2, w_3, \ldots$ may not be very important, their flow exerts a strong influence on the running coupling in this approximation.

Together with the inclusion of Z_k in the regulator, our estimate for the β function would be of the form $\beta_{g^2} = \eta g^2$ with

$$\eta = -\frac{b_0 \frac{g^2}{(4\pi)^2} + b_1 \frac{g^2}{(4\pi)^2} w_2}{1 + d_1 \frac{g^2}{(4\pi)^2} + d_2 \frac{g^2}{(4\pi)^2} w_2}, \tag{147}$$

with a further coefficient d_2, and $d_1 < 0$ can be read off from Eq. (143). Particularly this d_1 makes an important contribution to the two-loop β function coefficient, as discussed above. Contrary to Eq. (146), this equation contains information to all orders in g^2, even for the strict truncation $w_2 = 0$, solely due to the spectral adjustment of the regulator.

Of course, for more complicated truncations, the deformation of the spectrum of $\Gamma_k^{(2)}$ becomes much more involved. A natural generalization would thus be the inclusion of the full $\Gamma_k^{(2)}$ in the regulator. However, any dependence of R_k on the fluctuation would invalidate the derivation of the flow equation in Eq. (18), especially spoiling the one-loop structure. But within the background formalism, we can at least include $\bar{\Gamma}_k^{(2)}$ where we have set all field dependence of Γ equal to the background field $\bar{A}$,

$$R_k(\bar{\Gamma}_k^{(2)}) = \bar{\Gamma}_k^{(2)}\, r(\bar{\Gamma}_k^{(2)}/(Z_k k^2), \tag{148}$$

such that the regulator is fully adjusted to the spectral flow of the fluctuation operator evaluated at the background field. The resulting flow equation then contains also $\bar{\Gamma}_k^{(2)}$ derivatives; for instance, at $A = \bar{A}$ where $\Gamma_k^{(2)} = \bar{\Gamma}_k^{(2)}$, we find

$$\partial_t \Gamma_k = \frac{1}{2}\,\mathrm{STr}\left[(2-\eta)\,\frac{-y\,r'(y)}{1+r(y)} + \frac{\partial_t \Gamma_k^{(2)}}{\Gamma_k^{(2)}}\,\frac{r(y)+yr'(y)}{1+r(y)}\right]_{y=\frac{\Gamma_k^{(2)}}{Z_k k^2}}. \tag{149}$$

Despite the additional terms, this form of the flow equation (together with the approximation of setting $A = \bar{A}$ already for finite k) is technically advantageous, since it can be written in generalized propertime form by means of a Laplace transformation, see the discussion in the preceding section.

The additional $\partial_t \Gamma_k^{(2)}$ terms are the generalization of the η term in Eq. (141) and thus can support a further resummation of a large class of diagrams. In [78], this has been used together with an operator-expansion truncation involving arbitrary powers of the field strength invariant $(F^a_{\mu\nu}F^a_{\mu\nu})^n$, $n = 1, 2, \ldots$, to get a nonperturbative estimate of the β function. The anomalous dimension has the structure,

$$\eta = -\frac{b_0\,\frac{g^2}{(4\pi)^2} + b_1\,\frac{g^4}{(4\pi)^4} + b_2\,\frac{g^6}{(4\pi)^6} + \ldots}{1 + d_1\,\frac{g^2}{(4\pi)^2} + d_2\,\frac{g^4}{(4\pi)^4} + d_3\,\frac{g^6}{(4\pi)^6} + \ldots}, \tag{150}$$

the form of which can approximately be resummed in a closed-form integral expression, see [78]. The resulting β function exhibits a second zero at $g^2 \to g_*^2 > 0$, corresponding to an IR fixed point, see Fig. 10. Hence the flow-equation results for the IR running coupling in background gauge based on an operator expansion show strong similarities to those in the Landau gauge based on a vertex expansion mentioned in Subsect. 3.5. Within low-order perturbation theory, the universality of the running coupling is a well-known property. Independently of the different definitions of the coupling, the one-loop (and in mass-independent schemes also the two-loop) β-function coefficient is always the same. Here, we also observe a qualitative agreement between the Landau gauge and the background gauge in the nonperturbative IR in the form of an attractive fixed point. This points to a deeper connection between the two gauges which deserves further study and may be traced back to certain non-renormalization properties in the two gauges.

The background formalism has also been used for finite-temperature studies of Yang-Mills theory and the approach to chiral symmetry breaking, [86, 87], and for a study of nonperturbative renormalizability of gauge theories with extra dimensions [88]. The background-field formalism lies also at

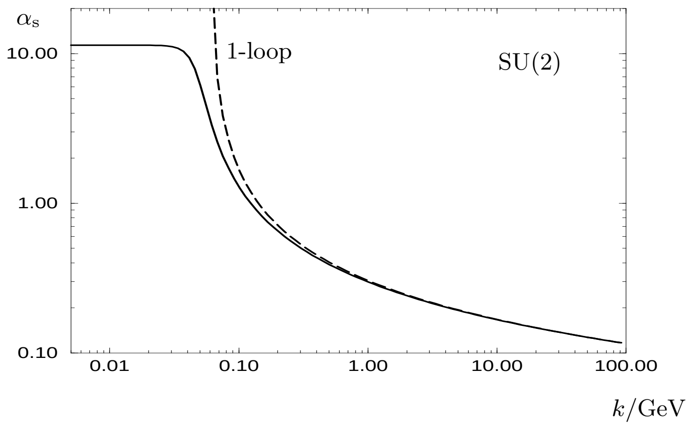


Fig. 10. Running coupling α_s versus momentum scale k in GeV for gauge group SU(2), using the initial value $\alpha_s(M_Z) \simeq 0.117$ for illustration. The solid line represents the result of an infinite-order resummation of Eq. (150) as taken from [78] in comparison with one-loop perturbation theory (dashed line).

the heart of a series of RG flow-equation studies of quantum gravity [89]; it is, of course, also useful for the study of abelian gauge theories such as the abelian Higgs model [90] or strong-coupling QED [91]. An alternative strategy to exploit a background field in RG flow equations has been proposed in [92]. Finally, the spectral adjustment of the regulator described here is a special case of the general strategy of functional optimization [8], cf. Subsect. 2.4.

5 From Microscopic to Macroscopic Degrees of Freedom

A typical feature of strongly interacting field theories is given by the fact that macroscopic degrees of freedom can be very different from microscopic degrees of freedom. For instance in QCD, quarks and gluons represent the microscopic degrees of freedom, whereas macroscopic degrees of freedom are mesons and baryons. The latter are bound states of quarks and gluons. Prominent representatives are the light pseudo-scalar mesons (pions, kaons, ...) which carry bi-fermionic quantum numbers, $\phi \sim \bar{\psi}\psi$. This type of fermionic pairing occurs in many systems, also in condensed-matter physics with strongly correlated electrons. Generically, a strong fermionic (self-)interaction is required for this pairing. In QCD, this is, of course, induced via the interactions with gluons. For an efficient description of the physics, it is advisable to take this transition from microscopic to macroscopic degrees of freedom into account [93, 94, 95, 96, 97, 98].

5.1 Partial bosonization

An explicit example for a fermion-to-boson transition is provided by the Hubbard-Stratonovich transformation, or *partial bosonization*. Let us discuss this transformation with the aid of a specific system: the Nambu–Jona-Lasinio (NJL) model [99]. We consider a version with one Dirac fermion, defined by the action

$$S_{\mathrm{F}} = \int \mathrm{d}^4x \left\{ \bar{\psi}\mathrm{i}\partial\!\!\!/\psi + \frac{1}{2}\,\lambda[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2] \right\}. \tag{151}$$

This model has a $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetry which here plays a similar role as chiral symmetry in QCD; in particular, it protects the fermions against acquiring a mass due to fluctuations.

Partial bosonization is obtained with the aid of the following mixed fermionic-bosonic theory,

$$S_{\mathrm{FB}} = \int \mathrm{d}^4x \left\{ \bar{\psi}\mathrm{i}\partial\!\!\!/\psi + m^2\phi^*\phi + h[\bar{\psi}P_{\mathrm{L}}\phi\psi - \bar{\psi}P_{\mathrm{R}}\phi^*\psi] \right\}, \tag{152}$$

with $P_{\mathrm{L,R}} = (1/2)(1 \pm \gamma_5)$ being the projectors onto left- and right-handed components of the Dirac fermion. In fact, the model (152) is equivalent to that of (151) also on the quantum level if

$$m^2 = \frac{h^2}{2\lambda}. \tag{153}$$

For a proof, it suffices to realize that

$$\begin{aligned}
\int \mathcal{D}\phi\,\mathrm{e}^{-S_{\mathrm{FB}}} &= \mathrm{e}^{-\int \bar{\psi}\mathrm{i}\partial\!\!\!/\psi} \int \mathcal{D}\phi\,\mathrm{e}^{-\int\left(\phi^* + \frac{h}{m^2}\bar{\psi}P_{\mathrm{L}}\psi\right)m^2\left(\phi - \frac{h}{m^2}\bar{\psi}P_{\mathrm{R}}\psi\right)} \\
&\qquad\qquad \times \mathrm{e}^{-\int \frac{h^2}{m^2}(\bar{\psi}P_{\mathrm{L}}\psi)(\bar{\psi}P_{\mathrm{R}}\psi)} \\
&= \mathcal{N}\,\mathrm{e}^{-\int\left(\bar{\psi}\mathrm{i}\partial\!\!\!/\psi + \frac{1}{4}\frac{h^2}{m^2}[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2]\right)} \overset{(153)}{\equiv} \mathcal{N}\,\mathrm{e}^{-S_{\mathrm{F}}},
\end{aligned} \tag{154}$$

where we have used that $(\bar{\psi}P_{\mathrm{L}}\psi)(\bar{\psi}P_{\mathrm{R}}\psi) = (\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2$, and $\mathcal{N}$ abbreviates the Gaußian integral over ϕ which is a pure number and can be absorbed into the normalization of the remaining fermionic integral $\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\,\mathrm{e}^{-S_{\mathrm{F}}}$. Also on the classical level, the equations of motion display the fermionic pairing, i.e., bosonization,

$$\phi = \frac{h}{m^2}\,\bar{\psi}P_{\mathrm{R}}\psi, \quad \phi^* = -\frac{h}{m^2}\,\bar{\psi}P_{\mathrm{L}}\psi. \tag{155}$$

Equation (152) is the starting point for mean-field theory. The fermionic integral is Gaußian now,

$$\int \mathcal{D}\phi\mathcal{D}\bar{\psi}\mathcal{D}\psi\,\mathrm{e}^{-S_{\mathrm{FB}}} = \int \mathcal{D}\phi\,\mathrm{e}^{-S_{\mathrm{B}}}, \tag{156}$$

resulting in the purely bosonic action

$$S_{\mathrm{B}} = \int \mathrm{d}^4 x \left\{ m^2 \phi^* \phi - \ln \det [\mathrm{i}\partial\!\!\!/ + h(P_{\mathrm{L}}\phi - P_{\mathrm{R}}\phi^*)] \right\}. \tag{157}$$

Mean-field theory now neglects bosonic fluctuations and assumes that the bosonic ground state corresponds to that of the classical bosonic action. Of course, the ln det is still a complicated nonlinear and nonlocal expression; nevertheless, assuming that the ground state is homogeneous in space and time, ϕ =const., the determinant can be computed by standard means [23]. For our purposes, it suffices to know that for

$$\lambda > \frac{8\pi^2}{\Lambda^2} \equiv \lambda_{\mathrm{cr}}, \tag{158}$$

with Λ being the UV cutoff, the resulting effective potential $V_{\mathrm{B}}(\phi^*\phi)$ has a nonzero minimum, implying a nonzero vacuum expectation value $\langle\phi\rangle \neq 0$. In the fermionic language, this vacuum expectation value corresponds to a bi-fermionic condensate $\langle\phi\rangle \sim \langle\bar{\psi}\psi\rangle$ (a chiral condensate in the QCD context). The expectation value generates fermion mass terms $\sim m_{\mathrm{f}}\bar{\psi}\gamma_5\psi$ with[10]

$$m_{\mathrm{f}} \sim h\langle\phi\rangle, \tag{159}$$

and the U(1)×U(1) symmetry is spontaneously broken to U(1) (fermion number conservation); this implies the existence of one Goldstone boson corresponding to excitations of the phase of the ϕ field. In the QCD context, this scenario corresponds to the spontaneous break-down of chiral symmetry with the pseudo-scalar mesons as Goldstone bosons. For $\lambda < \lambda_{\mathrm{cr}}$, the vacuum expectation value is zero, $\langle\phi\rangle = 0$ and the system remains in the symmetric phase.

These models of NJL type with broken symmetry show many similarities with QCD phenomenology, but there are important caveats from a microscopic viewpoint. First, there is no microscopic four-quark (or higher) self-interaction beyond criticality in QCD, $\lambda|_\Lambda \to 0$. In other words, $\lambda[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2]$ and other four-fermion operators are RG irrelevant; the Euclidean microscopic action for vanishing current quark masses is given by

$$S_{\mathrm{QCD}} = \int \mathrm{d}^D x \left(\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \mathrm{i}\bar{\psi} D\!\!\!\!/\,\psi \right), \tag{160}$$

where $D_\mu = \partial_\mu + \mathrm{i}\bar{g}\tau^c A^c_\mu$ denotes the gauge covariant derivative in the fundamental representation, which is generated by the τ^a with $\mathrm{tr}[\tau^a\tau^b] = (1/2)\delta^{ab}$. Of course, four-quark operators in the effective action are generated by gluon exchange from fluctuations described by box diagrams; see Fig. 11. The resulting β function for the four-quark coupling reads to lowest order

[10] The occurrence of γ_5 in the fermion mass term arises from our fermion conventions [100]; these are related to more standard conventions by a discrete chiral rotation.

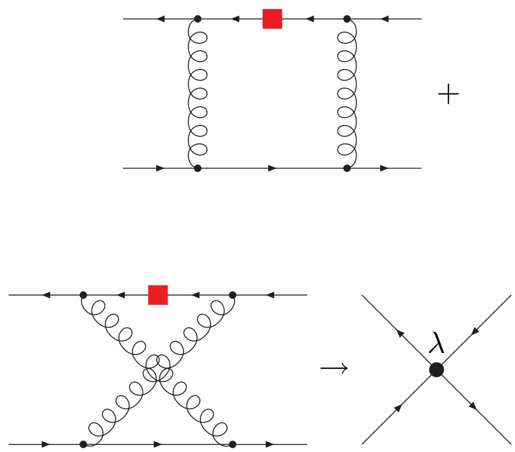


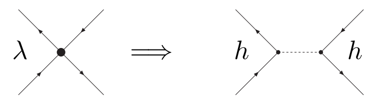
Fig. 11. Box diagrams with fundamental QCD interactions generate effective four-fermion self-interactions interactions λ. (Only one diagram per topology is shown; further diagrams with the regulator insertion (filled box) attached to other internal lines, of course, also exist.) The resulting flow-equation contribution to the running of λ is given in Eq. (161).

$$\partial_t \lambda \equiv \beta_\lambda = -c_\lambda \frac{1}{k^2} g^4, \tag{161}$$

where c_λ is a coefficient which depends on the algebraic structure of the theory and the details of the IR regularization. For definiteness, let us consider QCD with an SU(3) gauge sector but only one massless quark flavor $N_{\mathrm{f}} = 1$, the classical action of which has the same "chiral" symmetry properties as the NJL model used above. For this system, the coefficient c_λ obtains $c_\lambda = 5/(12\pi^2) > 0$ for the regulator (33) and a Fierz decomposition as chosen in [101]. Obviously, the four-quark self-interaction λ is asymptotically free as it should be for a QCD-like theory. Of course, for increasing gauge coupling g towards the IR, a naive extrapolation of Eq. (161) predicts that the fermionic self-interaction can become critical $\lambda > \lambda_{\mathrm{cr}}$ for some IR scale k_{cr}. If this holds also in the full theory, the quark self-interaction $\lambda[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2]$ becomes strongly RG relevant at scales below k_{cr} and we expect the system to end up in the symmetry-broken phase [102].

This scenario appears to match our expectations for QCD. But in order to arrive at a quantitative description we do not only face the problem of computational control in the nonperturbative domain; moreover, we have to deal with the conceptual problem of how to switch from the description in terms of quarks and gluons to that involving boson fields as well. In view of the Hubbard-Stratonovich transformation, we may be tempted to apply partial bosonization at some scale $k_{\mathrm{B}} < k_{\mathrm{cr}}$. However, it turns out that this leads to a strong spurious dependence on the precise choice of k_{B} in generic truncations. This has to do with the following observation. Let us naively partially bosonize at $k_{\mathrm{B}} < k_{\mathrm{cr}}$. Diagrammatically,

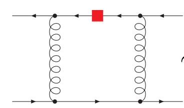
$k_{\mathrm{B}}:$



$$S_{\mathrm{F}} \quad \Longrightarrow \quad S_{\mathrm{FB}}, \tag{162}$$

where $\lambda = 0$ in S_{FB} after bosonization. Now, let us perform another RG step and integrate out another momentum shell Δk. Owing to the box diagrams, new quark self-interactions are generated again in this RG step,

$k - \Delta k: \quad \partial_t \lambda =$



$$\sim g^4 \neq 0. \tag{163}$$

In other words, the bosonizing field ϕ which did a perfect job at k_{B} is no longer perfect at $k - \Delta k$; it does no longer bosonize all quark self-interactions which it was introduced for. Incidentally, this problem does not only occur if gauge interactions are present, as Eq. (163) seems to suggest. The same problem arises, e.g., in purely fermionic systems where fermion self-interactions are generated by the flow in many different channels also with nonlocal structure; since partial bosonization with a local bosonic interaction can never account for all four-fermion vertices, the remaining four-fermion structure will again generate the full structure in the RG flow.

In a regime where many couplings run fast, neglecting the newly generated terms (163) can introduce large errors. But keeping these terms seems to make partial bosonization redundant. The solution of this dilemma is the subject of the following subsection.

5.2 Scale-dependent field transformations

In Eqs. (162),(163), we have observed that different bosonizing fields are needed to compensate the quark self-interactions at different scales. This points already to a solution of the problem [94]: we promote the bosonizing field ϕ to be scale dependent, $\phi \to \phi_k$, the flow of which can be written as

$$\partial_t \phi_k = \mathcal{C}_k[\phi, \psi, \bar{\psi}, \ldots]. \tag{164}$$

Here, $\mathcal{C}_k$ is an a priori arbitrary functional of possibly all fields in the system. For the present problem, the idea is to choose $\mathcal{C}_k$ such that the resulting effective action $\Gamma_k[\phi_k]$ does not possess fermionic self-interactions. For more general cases, the functional $\mathcal{C}_k$ can be chosen such that the effective action $\Gamma_k[\phi_k]$ becomes simple, since its simplicity is a strong criterion for the proper choice of the relevant degrees of freedom.

We can formulate this idea in a differential fashion: we are looking for a functional $\mathcal{C}_k$ which yields a flow of $\Gamma_k[\phi_k]$ taken at fixed ϕ_k, (suppressing further field dependencies on $\bar{\psi}, \psi, \ldots$),

$$\partial_t \Gamma_k[\phi_k]|_{\phi_k} = \partial_t \Gamma_k[\phi_k] - \int \frac{\delta \Gamma_k[\phi_k]}{\delta \phi_k} \, \partial_t \phi_k, \tag{165}$$

such that the flow of the fermion self-interaction vanishes for all k, $\partial_t \lambda|_{\phi_k} = 0$. Therefore, if $\lambda = 0$ holds at one scale k, λ stays zero at all scales, and ϕ_k becomes the "perfect" boson an all scales. The right-hand side of Eq. (165) can be read as follows: the first term denotes the full RG flow given in terms of a flow equation, whereas the second term with $\partial_t \phi_k = \mathcal{C}_k$ characterizes how ϕ_k has to be modified scale by scale in order to partially bosonize fermion self-interactions on all scales. This fixes the functional form of $\mathcal{C}_k$.

Now, we need a flow equation for the effective action $\Gamma_k[\phi_k]$ with scale-dependent field variables. This can indeed be formulated in various ways. Here, we follow a general and flexible exact construction given in [8]. Consider the modified generating functional

$$Z_k[J] = \mathrm{e}^{W_k[J]} = \int \mathcal{D}\varphi \, \mathrm{e}^{-S[\varphi] - \frac{1}{2} \int \varphi_k R_k \varphi_k + \int J \varphi_k}, \tag{166}$$

where we have coupled a scale-dependent φ_k to the source and the regulator. This combination guarantees that the resulting flow equation has a one-loop structure [29]. The scale dependence of φ is given by

$$\partial_t \varphi_k = \tilde{\mathcal{C}}_k[\varphi], \tag{167}$$

similar to Eq. (164) with the difference that Eq. (167) is formulated under the functional integral, whereas Eq. (164) holds for the fields conjugate to the source J,

$$\phi_k = \langle \phi_k \rangle \equiv \frac{\delta W_k[J]}{\delta J}. \tag{168}$$

Hence, $\mathcal{C}_k$ and $\tilde{\mathcal{C}}_k$ generally are not identical.

The derivation of the flow of $W_k[J]$ is straightforward:

$$\begin{aligned} \partial_t W_k[J] &= \frac{1}{Z_k[J]} \int \mathcal{D}\varphi \left(J \partial_t \varphi_k - \frac{1}{2} \int \varphi_k \partial_t R_k \varphi_k - \int \varphi_k R_k \partial_t \varphi_k \right) \\ &\qquad \times \mathrm{e}^{-S[\varphi] - \Delta S_k[\varphi_k] + \int J \varphi_k} \\ &= J \langle \partial_t \varphi_k \rangle - \frac{1}{2} \mathrm{Tr} \partial_t R_k G_k - \int \frac{\delta}{\delta J} R_k \langle \partial_t \varphi_k \rangle \\ &\quad - \int \phi_k R_k \langle \partial_t \varphi_k \rangle - \frac{1}{2} \int \phi_k \partial_t R_k \phi_k. \end{aligned} \tag{169}$$

Here, we have defined the propagator similar to Eq. (19)

$$G_k(p) = \left(\frac{\delta^2 W_k}{\delta J \delta J}\right)(p) = \langle \varphi_k(-p)\varphi_k(p)\rangle - \phi_k(-p)\phi_k(p). \tag{170}$$

We have also used the relation $\langle \varphi_k \partial_t \varphi_k \rangle = (\frac{\delta}{\delta J} + \phi_k)\langle \partial_t \varphi_k\rangle$. As usual, we define the effective action by means of a Legendre transformation, this time involving the scale-dependent field variables (cf. Eq. (20)) ,

$$\Gamma_k[\phi_k] = \sup_J \left(\int J\phi_k - W_k[J]\right) - \frac{1}{2}\int \phi_k R_k \phi_k. \tag{171}$$

The resulting flow of this effective action is

$$\partial_t \Gamma_k[\phi_k] = \frac{1}{2}\mathrm{Tr}\, \partial_t R_k G_k + \int \left(G_k \frac{\delta}{\delta \phi_k}\right) R_k \langle \partial_t \varphi_k\rangle + \int \frac{\delta \Gamma_k}{\delta \phi_k}(\partial_t \phi_k - \langle \partial_t \varphi_k\rangle), \tag{172}$$

where, in the course of the Legendre transformation, all J dependence turns into a ϕ_k dependence by virtue of $J = J_{\mathrm{sup}} = J[\phi_k]$; this also implies $\frac{\delta}{\delta J} = G_k \frac{\delta}{\delta \phi_k}$.

For a given scale-dependent field transformation $\partial_t \varphi_k = \tilde{\mathcal{C}}_k[\varphi]$, we can successively work out $\langle \partial_t \varphi_k\rangle$, $\phi_k = \langle \varphi_k \rangle$ and $\partial_t \phi_k = \partial_t \langle \varphi_k\rangle$; in general, the latter is not identical to $\langle \partial_t \varphi_k\rangle$. Following this strategy, we would only then be able to compute the flow of $\Gamma_k[\phi_k]$. Also $\mathcal{C}_k$ would be a derived quantity, fixed implicitly by the choice of $\tilde{\mathcal{C}}_k$.

By contrast, we can supplement the flow equation (172) with a bootstrap argument: since all we want to choose in the end is $\partial_t \phi_k = \mathcal{C}_k[\phi]$, the precise form of $\partial_t \varphi = \tilde{\mathcal{C}}_k$ need not be known; in fact, $\tilde{\mathcal{C}}_k$ does not occur directly in Eq. (172), but only in expectation values. Therefore, we simply assume that a suitable $\tilde{\mathcal{C}}_k$ exists for a desired $\mathcal{C}_k$ such that

$$\langle \partial_t \varphi_k \rangle \stackrel{!}{=} \partial_t \phi_k. \tag{173}$$

Of course, this is a highly implicit construction, and in view of the complicated structure of the mapping $\tilde{\mathcal{C}}_k \to \mathcal{C}_k$, the existence of a suitable $\tilde{\mathcal{C}}_k$ for an arbitrary $\mathcal{C}_k$ is generally not guaranteed or, at least, difficult to prove. Nevertheless, since the resulting flow equation will, in practice, be used together with a truncation, it is reasonable to assume that Eq. (174) can at least be satisfied to the order of the truncation. As a consequence of Eq. (174), the flow equation simplifies,

$$\begin{aligned} \partial_t \Gamma_k[\phi_k]|_{\phi_k} &\stackrel{(165)}{=} \partial_t \Gamma_k[\phi_k] - \int \frac{\delta \Gamma_k[\phi_k]}{\delta \phi_k}\, \partial_t \phi_k \\ &= \frac{1}{2}\mathrm{Tr}\, \partial_t R_k G_k + \int \left(G_k \frac{\delta}{\delta \phi_k}\right) R_k \partial_t \phi_k - \int \frac{\delta \Gamma_k[\phi_k]}{\delta \phi_k}\, \partial_t \phi_k, \end{aligned} \tag{174}$$

which is the desired flow equation for scale-dependent field variables. Apart from the standard first term $\sim \mathrm{Tr} \partial_t R_k G_k$ and the third term which carries the explicit scale dependence of ϕ_k, we encounter the second term which takes care of fluctuation contributions to the renormalization flow of the operator insertion $\partial_t \varphi_k$ in the functional integral. Actually, this second term will generally be subdominant for not too large coupling: first, it is of higher order in the coupling, and second, R_k insertions lead to weaker numerical coefficients than $\partial_t R_k$ insertions for standard regulators. We expect that this term does not induce strong modifications for couplings up to $\mathcal{O}(1)$.

5.3 Scale-dependent field transformations for QCD: Rebosonization

Let us now turn back to our original problem of QCD-like systems, where quark self-interactions are generated by gluon exchange. In order to arrive at a mesonic description in the infrared, we now want to apply scale-dependent field transformations that translate the quark self-interactions into the bosonic sector an all scales – a process that may be termed partial *rebosonization.*

In order to illustrate the formalism, let us study one-flavor QCD in a simple truncation. Apart from the standard kinetic terms for the quark and gluons, supplemented by wave function renormalization factors, we include a point-like four-quark self-interaction in the scalar–pseudo-scalar sector[11]

$$\Gamma_{\mathrm{F},k} = \int \mathrm{d}^4 x \left(\frac{Z_k}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \mathrm{i} Z_\psi \bar{\psi} \not{D} \psi + \frac{1}{2}\, \lambda [(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2] \right). \qquad (175)$$

The initial condition of the four-quark operator in the UV is obviously given by $\lambda|_{k\to\Lambda} \to 0$. As a first step towards rebosonization, we include a complex mesonic scalar field in the truncation on equal footing,

$$\Gamma_k = \Gamma_{\mathrm{F},k} + \int \mathrm{d}^4 x \left(Z_\phi \partial_\mu \phi^* \partial_\mu \phi + V(\phi^*\phi) + h[\bar{\psi} P_{\mathrm{L}} \phi \psi - \bar{\psi} P_{\mathrm{R}} \phi^* \psi] \right), \qquad (176)$$

with a scalar potential $V(\phi^*\phi) = m^2 \phi^*\phi + \mathcal{O}((\phi^*\phi)^2)$. The initial conditions for the scalar field at $k \to \Lambda$ need to be chosen such that the scalar has no observable effect on the QCD sector whatsoever. This is easily done by demanding that the Yukawa interaction with the quarks vanishes $h|_{k\to\Lambda} \to 0$. We also choose a large scalar mass $m^2|_{k\to\Lambda} = \mathcal{O}(\Lambda^2)$, ensuring a fast decoupling of the scalar; finally, we set $Z_\phi|_{k\to\Lambda} \to 0$ which makes the scalar non-dynamical at the UV scale. Solving the flow with these initial conditions,

[11] Of course, in order to avoid any ambiguity with respect to possible Fierz rearrangements of the four-fermion interactions in the point-like limit, all possible linearly-independent four-fermion interactions, in principle, have to be included in the truncation. For simplicity, we confine ourselves here just to the scalar–pseudo-scalar channel, where chiral condensation is expected to occur. For the four-fermion interactions that will be generated by the flow, we use the Fierz decomposition as proposed in [101].

the scalars rapidly decouple and only the standard QCD flow remains, revealing the purely formal character of this first step towards rebosonization.

As a second step, we now use the freedom to perform scale-dependent field transformations, as suggested in Eq. (164). We promote the field ϕ to be scale dependent, and choose the functional $\mathcal{C}_k$ characterizing this scale dependence to be of the form

$$\partial_t \phi_k = \bar{\psi} P_{\rm R} \psi \, \partial_t \alpha_k, \quad \partial_t \phi_k^* = -\bar{\psi} P_{\rm L} \psi \, \partial_t \alpha_k, \tag{177}$$

with some function α_k to be determined below. At this stage, let us study the consequences of the last term in Eq. (174) on the resulting flow; note that we are now dealing with a complex field, such that this term goes over into, $\int \frac{\delta \Gamma_k}{\delta \phi_k} \partial_t \phi_k \to \int \frac{\delta \Gamma_k}{\delta \phi_k} \partial_t \phi_k + \int \frac{\delta \Gamma_k}{\delta \phi_k^*} \partial_t \phi_k^*$. From the Yukawa interaction, we get from this term, together with Eq. (177),

$$h[\bar{\psi} P_{\rm L} \phi \psi - \bar{\psi} P_{\rm R} \phi^* \psi] \to \frac{1}{2} h \partial_t \alpha_k [(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2], \tag{178}$$

which is a contribution to the flow of the quark self-interaction. Together with contributions from the first term of Eq. (174), and neglecting the second term of Eq. (174) here and in the following as discussed above, we obtain the flow of the four-quark coupling at fixed ϕ_k,

$$\partial_t \lambda|_{\phi_k} = -c_\lambda \, \frac{1}{k^2} \, g^4 - h \, \partial_t \alpha_k, \tag{179}$$

where we used the result given in Eq. (161). Choosing the transformation function

$$\partial_t \alpha_k = -c_\lambda \, \frac{1}{k^2} \, \frac{g^4}{h}, \tag{180}$$

we obtain

$$\partial_t \lambda|_{\phi_k} = 0, \tag{181}$$

which, together with the initial condition $\lambda|_{k\to\Lambda} \to 0$ implies that $\lambda = 0$ holds for all scales k. The scale-dependent transformation Eq. (177) thus has removed the point-like four-quark interaction in the scalar–pseudo-scalar sector completely by partial rebosonization. The information about this interaction is transformed into the scalar sector; for instance, the scalar mass term is also subject to the last term of Eq. (174):

$$m^2 \phi^* \phi \to -m^2 \partial_t \alpha_k [\bar{\psi} P_{\rm L} \phi \psi - \bar{\psi} P_{\rm R} \phi^* \psi]. \tag{182}$$

This yields a contribution to the flow of the Yukawa coupling. Together with the contribution from the first term of Eq. (174), i.e., the standard flow term, we obtain

$$\begin{aligned} \partial_t h|_{\phi_k} &= -\frac{1}{2} \, c_h \, g^2 h + m^2 \, \partial_t \alpha_k \\ &\overset{(180)}{=} -\frac{1}{2} \, c_h \, g^2 h - c_\lambda \, \frac{m^2}{k^2} \, \frac{g^4}{h}, \end{aligned} \tag{183}$$

Fig. 12. Diagrams contributing to the RG flow of the scalar sector: (a) flow of the Yukawa coupling, see Eq. (183); (b) flow of the scalar mass, see Eq. (184). Only one diagram per topology is shown; further diagrams exhibit the regulator insertion (filled box) attached to other internal lines.

where the coefficient c_h is a result of the diagram shown in Fig. 12(a). For the linear regulator and in the Landau gauge, the result is $c_h = 1/\pi^2$ for SU(3). The second term of Eq. (183) accounts for the fact that the fermions couple to a scale-dependent boson. The flow of the scalar mass term is not transformed by the choice of Eq. (177); only the standard flow-equation term contributes,

$$\partial_t m^2 = c_m\, k^2\, h^2, \tag{184}$$

where the coefficient c_m yields for the regulator (33) $c_m = N_\mathrm{c}/(8\pi^2)$, resulting from the diagram in Fig. 12(b). The physical properties of the resulting boson field can best be illustrated with the convenient dimensionless composite coupling

$$\tilde{\epsilon} := \frac{m^2}{k^2 h^2}, \tag{185}$$

and its β function

$$\partial_t \tilde{\epsilon} = -2\tilde{\epsilon} + c_m + c_h g^2 \tilde{\epsilon} + 2 c_\lambda g^4 \tilde{\epsilon}^2, \tag{186}$$

resulting from Eqs. (161),(183),(184). The last term comes directly from rebosonization. Since all $c_i > 0$, this β function looks like a parabola; see Fig. 13, solid line. Without rebosonization, this β function would have corresponded to a straight line, see Fig. 13, dashed line.

The β function Eq. (186) exhibits two fixed points: $\tilde{\epsilon}_{*,1}$ is IR repulsive and $\tilde{\epsilon}_{*,2}$ is IR attractive. In a small-gauge-coupling expansion, the positions of the fixed points are given by

$$\tilde{\epsilon}_{*,1} \simeq \frac{c_m}{2} + \mathcal{O}(g^2), \quad \tilde{\epsilon}_{*,2} \simeq \frac{1}{c_\lambda g^4} + \mathcal{O}(1/g^2). \tag{187}$$

Without rebosonization, only $\tilde{\epsilon}_{*,1}$ is present.

If we start with initial conditions such that $\tilde{\epsilon}|_{k\to\Lambda} < \tilde{\epsilon}_{*,1}$, $\tilde{\epsilon}$ quickly becomes negative, corresponding to the bosonic mass term dropping below zero, $m^2 < 0$. This indicates that the potential develops a nonzero minimum, giving rise to chiral symmetry breaking and quark mass generation. However, we obtain

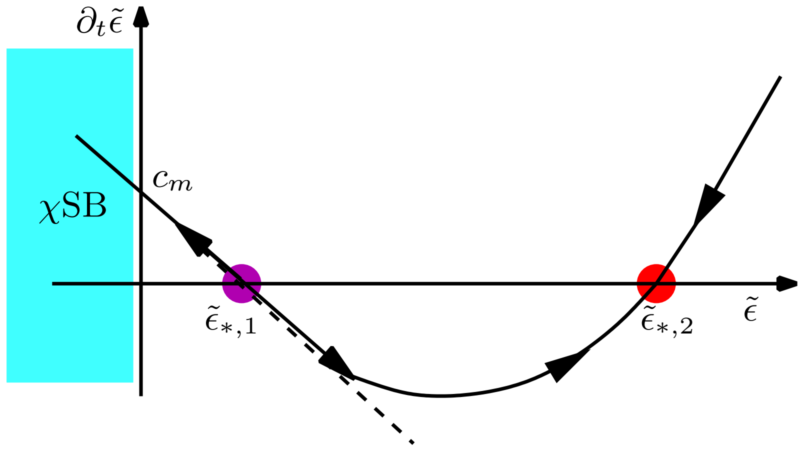


Fig. 13. Schematic plot of the β function (186) for the composite coupling $\tilde{\epsilon} = \frac{m^2}{k^2 h^2}$ with arrows pointing along the flow towards the IR. The fixed point $\tilde{\epsilon}_{*,1}$ is IR repulsive; in its vicinity, the scalar field behaves as a fundamental scalar (dashed line). If the flow is initiated with $\tilde{\epsilon}|_{k=\Lambda} < \tilde{\epsilon}_{*,1}$, $\tilde{\epsilon}$ drops quickly below zero and the system runs into the regime with chiral symmetry breaking (χSB). For $\tilde{\epsilon}|_{k=\Lambda} > \tilde{\epsilon}_{*,1}$, the system rapidly approaches the bound-state IR fixed point $\tilde{\epsilon}_{*,2}$, where the scalar exhibits bound-state behavior. QCD initial conditions correspond to $\tilde{\epsilon}|_{k\to\Lambda} \to \infty$.

this initial condition $\tilde{\epsilon}_{k\to\Lambda} < \tilde{\epsilon}_{*,1}$ only if either the scalar mass is small or the Yukawa coupling is large or both; but this is in conflict with our QCD initial conditions, specified below Eq. (176). With this initial condition, the system is not in the QCD universality class. Near $\tilde{\epsilon}_{*,1}$, the slope of the β function (186) is -2, which is nothing but a typical quadratic renormalization of a bosonic mass term; the scalar behaves like an ordinary fundamental scalar here. In fact, there is a correspondence between $\tilde{\epsilon}_{*,1}$ and the critical coupling of NJL-like systems,

$$\tilde{\epsilon}_{*,1} \simeq \frac{Nc}{2k^2 \lambda_{\rm cr}}, \tag{188}$$

cf. Eqs. (153),(158),(185); the factor of $N_{\rm c}$ takes care of the additional color degree of freedom of the quarks which was not present in the NJL system of Subsect. 5.1. The initial condition $\tilde{\epsilon}|_{k\to\Lambda} < \tilde{\epsilon}_{*,1}$ agrees with $\lambda|_{k\to\Lambda} > \lambda_{\rm cr}$, and the system is in the broken phase of the NJL model.

For $\tilde{\epsilon}|_{k\to\Lambda} > \tilde{\epsilon}_{*,1}$, the system quickly approaches $\tilde{\epsilon}_{*,2}$ either from above or below rather independently of the initial values of $m, h|_{k\to\Lambda}$. QCD initial conditions with large initial m and small initial h correspond to $\tilde{\epsilon}|_{k\to\Lambda} \to \infty$. But also for much smaller initial $\tilde{\epsilon}$, the system rapidly flows to $\tilde{\epsilon}_{*,2}$, and the memory of the precise initial values gets lost. There, the system is solely determined by the gauge coupling g^2 which governs the fixed-point position. This is precisely how it should be in QCD.

Near $\tilde{\epsilon}_{*,2}$, the boson is not really a fully developed degree of freedom. The flow does not at all remind us of the flow of a fundamental scalar, but points to the composite nature of the scalar. This justifies to call $\tilde{\epsilon}_{*,2}$ the bound-state fixed point; for instance, in weakly coupled systems such as QED, the boson at the bound-state fixed point describes a positronium-like bound state.

It is a particular strength of the RG approach with scale-dependent field transformations that one and the same field can describe bound state formation on the one hand and condensate formation as well as meson excitations on the other hand; whether the field behaves as a bound state or as a fundamental scalar is solely governed by the dynamics and the coupling strength of the system.

So far, our analysis of the system was essentially based on weak-coupling arguments, revealing that QCD at initial stages of the flow approaches the bound-state fixed point. But, we still have to answer a crucial question: how does QCD leave the bound-state fixed point and ultimately approach the chiral-symmetry broken regime? The answer is again given by the gauge coupling which controls the whole flow. For increasing gauge coupling, the parabola characterizing the β function (186) for $\tilde{\epsilon}$ is lifted, as depicted in Fig. 14. At a critical coupling value, $g^2 = g^2_{\mathrm{cr}}$, the fixed points $\tilde{\epsilon}_{*,1}$ and $\tilde{\epsilon}_{*,2}$ annihilate each other and the system runs towards the chiral-symmetry broken regime. This transition is unambiguously triggered by gluonic interactions. For instance in the present case of one-flavor QCD with gauge group SU(3), the critical coupling is given by $\alpha_{\mathrm{cr}} \equiv \frac{g^2_{\mathrm{cr}}}{4\pi} \simeq 0.74$ for the regulator Eq. (33). Since this coupling value is not a universal quantity, one should not overemphasize its meaning; however, it is interesting to observe that this coupling strength is in the nonperturbative domain, as expected, but not very deeply. In particular, since loop expansions go along with the expansion parameter α/π, the critical coupling and thus the approach to chiral symmetry breaking appears still to be in reach of weak-coupling methods (not to be confused with perturbation theory).

We conclude that the continuous scale-dependent translation allows for a controllable transition between microscopic to macroscopic degrees of freedom and between different dynamical regimes of a system. From a quantitative viewpoint, no spurious dependence on a bosonization scale, i.e., a scale at which degrees of freedom are discretely changed, is introduced, because field transformations are continuously performed on all scales. This helps maintaining the predictive power of truncated RG flows. As a result, macroscopic parameters can quantitatively be related to microscopic input.

5.4 Further reading: aspects of field transformations*

The scale-dependent field transformations introduced above were illustrated with the aid of n-point interactions in the point-like limit, e.g., a four-fermion interaction in the zero-momentum limit. Of course, the formalism can also be used if momentum dependencies of the vertices are taken into account. This

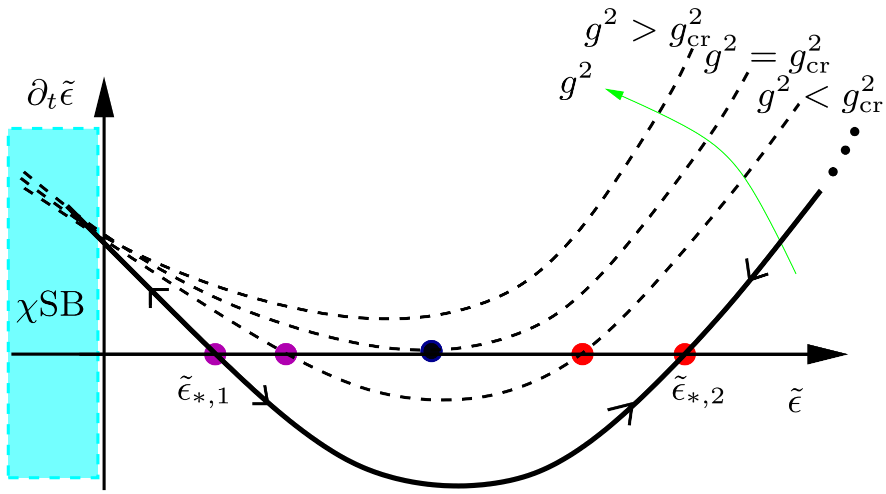


Fig. 14. Schematic plot of the β function (186) for the composite coupling $\tilde\epsilon = \frac{m^2}{k^2 h^2}$ with arrows pointing along the flow towards the IR. At weak gauge coupling, QCD-like systems first flow to the bound-state fixed point $\tilde\epsilon_{*,2}$ where they remain over a wide range of scales. For increasing gauge coupling g^2, the β function is lifted (dashed lines). At the critical coupling $g^2 = g^2_{\rm cr}$, the fixed points are destabilized and the system rapidly runs into the chiral symmetry broken regime (χSB).

is particularly important in the context of rebosonization, since composite bosonic fluctuations and bound states manifest themselves by a characteristic momentum dependence in the fermionic n-point correlators; for instance, a bosonic bound state corresponds to a pole in the s channel of the fermionic Minkowskian 4-point vertex.

A generic momentum dependence of an n-point vertex is nonlocal in coordinate space. By means of a field transformation, certain nonlocalities can be mapped onto a local description. The Hubbard-Stratonovich transformation is an example for such a transformation which maps a specific nonlocal four-fermion vertex onto a local bosonic theory with a local Yukawa coupling to the fermion.

With the following scale-dependent field transformation, this nonlocal-to-local mapping can be performed on all scales for a four-fermion vertex with s channel momentum dependencies, $\lambda = \lambda(s = q^2)$,

$$\partial_t \phi_k(q) = (\bar\psi P_{\rm R}\psi)(q)\,\partial_t\alpha_k(q), \quad \partial_t\phi^*_k(q) = -(\bar\psi P_{\rm L}\psi)(-q)\,\partial_t\alpha_k(q). \tag{189}$$

This results in a flow of the 4-point interaction for the transformed fields which reads

$$\partial_t\lambda|_{\phi_k} = \partial_t\lambda - h\,\partial_t\alpha_k(q). \tag{190}$$

Obviously, the q dependence of α_k can be chosen such the fluctuation-induced s-channel q dependence of λ is eaten up for all scales and all values of q.

Via the momentum-dependent analog of Eq. (183), this induces a momentum dependence of the Yukawa interaction $hb \to h(q)$; this can be taken care of by a further generalization of the field transformation,

$$\begin{aligned}\partial_t \phi_k(q) &= (\bar{\psi} P_{\mathrm{R}} \psi)(q)\, \partial_t \alpha_k(q) - \beta_k(q) \phi_k(q),\\ \partial_t \phi_k^*(q) &= -(\bar{\psi} P_{\mathrm{L}} \psi)(-q)\, \partial_t \alpha_k(q) - \beta_k(q) \phi_k^*(q).\end{aligned} \tag{191}$$

The resulting flows for the Yukawa coupling and the inverse scalar propagator for the transformed fields are then given by

$$\begin{aligned}\partial_t h(q)|_{\phi_k} &= \partial_t h(q) + \frac{Z_\phi q^2 + m^2}{h}\, \partial_t \lambda(q^2) + h\, \partial_t \beta_k(q),\\ (\partial_t Z_\phi(q) q^2 + \partial_t m^2)|_{\phi_k} &= \partial_t m^2 + 2 \partial_t \beta_k(q)\, (Z_\phi q^2 + m^2).\end{aligned} \tag{192}$$

For a given momentum dependence of λ, the first equation can be used to determine the transformation function $\beta_k(q)$.[12] The second equation then fixes the running of the mass and of the wave function renormalization of the transformed field. In this manner, specific nonlocal momentum structures can be transformed from the fermionic interactions into a local boson sector.

This strategy has been used for the abelian gauged NJL model [94], QED [91] and QCD-like systems in [101]. At weak coupling, one typically finds that the bound-state fixed point discussed above of the composite coupling $\tilde{\epsilon}$ has similar counterparts in many other couplings, such as the scalar dimensionless mass m^2/k^2 and the Yukawa coupling h. This is a manifestation of their RG irrelevance at weak coupling, with the dynamics of the scalar sector being fully controlled by the fermions and their gauge interactions. If the gauge coupling becomes large, the bound-state fixed points in all these couplings is destabilized and interactions involving composite bosons can become relevant or even dominant close to a transition into a broken-symmetry regime. Since this switching from irrelevant to relevant is a smooth process under the flow being controlled by the dynamics itself, the predictive power of the computation is maintained.

In QCD, the resulting effective action at IR scales naturally exhibits a sector which is similar to a chiral quark-meson model [103, 81, 104] but with all parameters fixed by the outcome of the RG flow with field transformations. Also the gluonic sector can still be dynamically active and contribute further to the running of the chiral sector.

A further application of the scale-dependent field transformation can be found in [105] where it is shown that the Fierz ambiguities mentioned in the preceding section can be overcome by treating all possible interaction channels and the corresponding scale-dependent bosonic composites on equal footing.

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[12] The momentum-independent part can, for instance, be fixed such that $\partial_t Z_\phi(q = k) = 0$, ensuring that the approximation of a momentum-independent Z_ϕ is self-consistent.

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